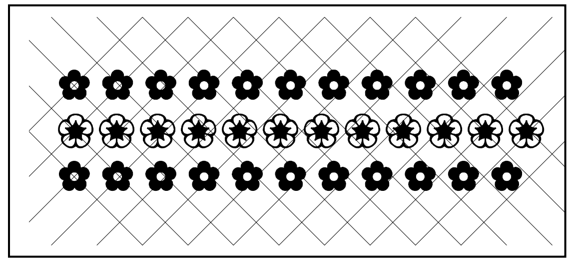
3-D Images.

You know those computer generated pictures that look like a scatter of dots but when you relax your eyes in just the right way a 3-D image pops out? Well, the idea behind those images is actually quite simple and leads to some nice geometry problems.

The first time I worked with this problem I was nervous because a lot of students told me that they had great trouble "seeing" those 3-D images. So I constructed the simple image at the right in hopes that they'd have an easier time with that, and that did turn out to be the case. I began by handing out copies of the image (reproduced in full at the end) and we spent some time getting as many people as possible to "get" the effect. In a class of 50 students, all but 5 managed to see the 3-D image. I also took in a supply of professional colour images (from a book called *Magic Eye*) that got traded around.



I asked the students to tell me what they "saw" and in fact two different versions of the image were described, one from those who relaxed their eyes and focused behind the page (the *divergent* case), and the other from those who "crossed" their eyes and focused in front of the page (the *convergent* case). Those who relaxed their eyes (the more common case) saw everything behind the page and saw the middle row of white flowers as being closest to them, then both rows of black flowers, and finally the grid as being the farthest away. Those who "crossed" their eyes saw everything in front of the page and saw the white flowers as being farthest away, then both rows of black flowers, and finally the grid as being the closest to them. Most students who saw it one way were unable see the other. A few could flip back and forth.

Then I gave them the problem. Suppose you were asked to build a 3-D model which would look to you exactly the same as the 3-D image you perceived. What would be the distance between the different levels: the row of white flowers, the rows of black flowers and the grid?

It's an interesting problem, in that in order to analyze it, you really have to understand exactly how the 3-D effect works. So the first half of the class is devoted to exploring that. I try to get the class to carry the conversation with minimal input from me, and to come up and draw the necessary illustrative diagrams (which appear below).

There are some basics to understand at the beginning. I ask what it is that allows us to judge how far away an object is? If we are gazing upon a scene, how do we tell which objects are closer and which are farther away? This question produces a number of different ideas.

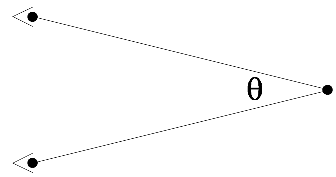
Often it takes a few moments to get the effect, and some final manoevering, backwards and forward, is necessary to get the image to appear sharp. Sometimes when you think you have it, you find that if you move your head back a bit you get a sharper image which is more comfortable to hold. Once you really get it, the effect is striking. By moving your eyes slightly from side to side, you can get the flowers moving relative to the grid in a way that is remarkably 3-dimensional.

The grid reminds me of a time when I was young watching a hockey game through a wire mesh fence that was about a meter in front of me, and suddenly the fence seemed much thicker and heavier than it had any right to be. I looked sharply at it and the effect disappeared instantly. But when I relaxed my eyes, the effect returned and I was astounded. What was really a feeble galvanized fence designed to stop flying pucks now looked like a heavy duty chain-link fencing around a prison yard. And it also seemed to be much farther away from me than I knew it was. [Why did the fence look heavier as well as farther away? Because we automatically judge the width of an object by integrating its apparent distance from us with the angle formed by its sides.]

Here's one. If you are allowed to move your head slightly from side to side, the relative apparent sideways motion of different objects will give you a clue as to which are closer. But that's not what's going on here. So what else?

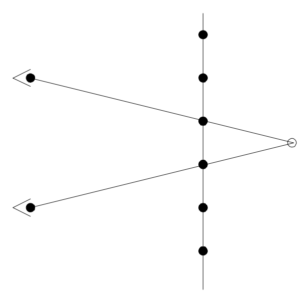
One student tells me that you use a different "focus" for objects at different distances. Indeed you do, and in that sense, your eye behaves exactly like the lens of a camera. Essentially, you adjust the curvature of the cornea which is the lens that focuses light from the object onto the retina. This is an adjustment that is available to someone with only one eye. But it is also not what is responsible for our 3-D effect.

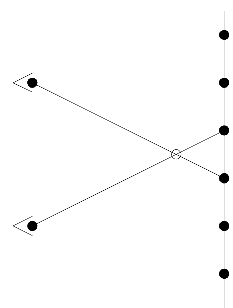
Finally the students start talking about angles. The mechanism behind the 3-D effect requires both eyes. It has to do with our ability to judge the angle between the line of sight of the object from each eye.



Study the diagram at the right. The light rays coming from the object to my two eyes form an angle and the nearer is the object, the larger is the angle. My brain is "aware" of the size of this angle, and this is the principal way in which it estimates the distance of the object.

With this concept firmly in place, I give the class 5 or 10 minutes of small group work to puzzle through the geometry of the 3-D phenomenon and promise them that I will invite them to present their argument. As I wander around the room, I am a bit surprised how difficult they are finding this. For example, I find some students examining the different form of the individual white and black flowers. They haven't seen that the effect relies on there being a *row* of flowers.

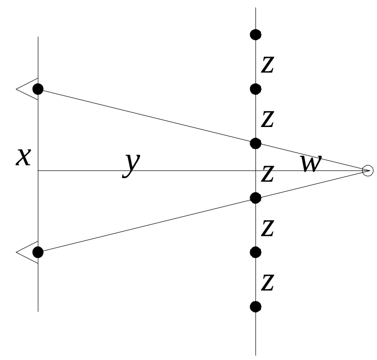




But most of the groups finally get it. And they produce for me two different diagrams for the two different ways of seeing. In both cases, the left eye focuses on one flower, and the right eye upon another, and the brain believes that these are the same flower. Now if that is the case—if they are the same flower—where must it be? It must be located at the point at which the two rays meet, and as the diagrams show, this will be either behind the page or in front of it. In the divergent case the right eye looks at the flower on the right and the left eye looks at the flower on the left and the brain tells you: *the flower is behind the page*. In the convergent case the right eye looks at the left flower and vice-versa, and your brain tells you: *the flower is in front of the page*.

The next step is to work out exactly how far in front of or behind the page it is. It takes a few minutes for the students to figure out how to approach this, but they soon realize that at least they are going to have to make some measurements.

For the divergent case the key distances are labeled on the diagram at the right. The distance between my eyes is x, the (perpendicular) distance from my eyes to the page is y, the distance between the flowers seen by the two eyes is z, and the distance from the page to the image is w. We want to find w, and we can measure x, y and z.



In fact let's work out the formula for w in terms of x, y and z. The two triangles in the diagram are similar. The big triangle has base x and altitude $y+w$ and the small triangle has base z and altitude w. Thus:

$$\frac{w}{z} = \frac{w+y}{x}$$

and this solves to give

$$wx = wz + yz$$

$$w = \frac{yz}{x-z} \ .$$

This is the calculation for the divergent case. The convergent case is left as an exercise..

Here is a set of typical measurements:

Between the eyes:	x = 60 mm.
Eyes to the page	y = 320 mm
Adjacent white flowers	z = 18 mm
Adjacent black flowers	z = 19 mm
Adjacent grid intersections	z = 20 mm

The distance between your eyes

If you are trying this out all on your own, a good measurement can be made by holding a ruler below your eyes and looking in a mirror. But don't forget to use only one eye at a time.

The formula then produces:

white flowers $\quad w = \dfrac{320\times 18}{60-18} = 137$ mm.

black flowers $\quad w = \dfrac{320\times 19}{60-19} = 148$ mm.

grid intersections $\quad w = \dfrac{320\times 20}{60-20} = 160$ mm.

On average the image is about 15 cm behind the page and the distance between the rows is just over 1 cm.

At this (late) stage a student asks: how do we know that the two eyes are looking at *adjacent* flowers? Maybe they are looking at flowers which are spaced two apart. Or even three.

That's an excellent question. How is it to be answered?

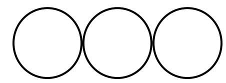
I get from the class a number of interesting ways to investigate this question. The simplest of these is this. Put a small dot in the white centre of one of the black flowers. Now when you are looking at the image, how many dots should you see (or rather be aware of)? The answer is two—one through the left eye and one through the right. Now here's the discriminating question: how far apart will those two dots be? If they appear to be in adjacent flowers, then you are juxtaposing adjacent flowers. If they appear to be two flowers apart, then you are juxtaposing flowers that are two flowers apart. I asked the students to try it out and in fact we got both answers!

We do the dot experiment and some students report that the dots are in adjacent flowers and others report that the dots are two flowers apart. So the two groups would get a different w value. In fact, not surprisingly, they also tended to have different y values (see Problem 1). Some students found that they could readily flip back and forth. In fact they said it was easier to get the images if they had the dot as a guide.

Problems

1. In the above Example, provide the calculation for the divergent case in which the juxtaposed flowers are two apart. Students found that the focus for this case was obtained with the page closer to the eyes than before. A typical measurement was y = 240 mm.
[Answer: for the grid, we get w = 480 mm.]

2. *Here's a lovely similar triangles problem.* I stand looking at a wall on which there are drawn three identical circles, their centres in a horizontal row, the middle circle tangent to each of the right and the left circle. I hold up a penny at a distance of 80 cm from my eyes. If I look through my left eye, the penny exactly covers the right circle, and if I look through my right eye, the penny exactly covers the left circle. How far am I from the wall? Assume my pupils are 6 cm apart and the penny is 2 cm in diameter.



3. In the above Example, provide the missing analysis for the convergent case. Show that the distance of the image *in front of* the page is $w = \dfrac{yz}{x+z}$. Those students who obtained the image this way tended to hold the page farther from the eyes than in the divergent case. A typical measurement was y = 480 mm. For the grid, show that the answer works out to be w = 120 mm (assuming adjacent flowers were juxtaposed).

4. This section was written one night while I was staying in Toronto at a friend's house (writing high school curriculum as chance would have it) and the following morning I found myself standing on an unfamiliar kitchen floor. It had a simple pattern of white squares formed by a square lattice of lines about 10 cm apart. I stood staring down at the floor, and because my mind was on these geometric things, I started adjusting my eyes. After a few moments of visual confusion, a new floor emerged and gradually sharpened into focus, but above my feet, and hovering around my waist. My impression was also that it was a finer, closer pattern than the real floor. I realized I must have crossed my eyes instead of relaxing them. There was now again the question of how deeply I had crossed them, so I focused again on the real floor and picked out a small dark spot in one of the squares. Then I recaptured the image and looked for the spot—there it lay, two copies of it, *two squares apart*. I estimated that my eyes were 150 cm above the floor. How far away from my eyes did the image floor appear to be? Assume my pupils are 60 mm apart.
[Answer: about 35 cm. Be careful with units.]

5. I stood once 50 cm away from a chain link fence and relaxed my eyes until the fence seemed much farther away. It also seemed to be made of much thicker metal; in fact under the sway of the optical illusion, I estimated the new thickness to be 3 times the actual thickness. If adjacent similar points on the grid were 1 cm apart, what was the distance between juxtaposed grid points? Take my eyes to be 6 cm apart.
[Answer: 4 cm.]