

Sum of Cubes

Jo was supposed to be studying for her grade 12 physics test, but her soul was wandering. *Show me something fun*, she said. Well I wasn't sure just what she had in mind, but it happened that I working on something (with a colleague and a couple of mathematicians in Georgia) that I thought was rather fun, so I pulled out a piece of paper, made two columns, and added them up:

<i>numbers</i>	<i>cubes</i>
1	1
2	8
3	27
4	64
10	100

I didn't say anything. *That's nice*, she said. Not a huge amount of enthusiasm, but enough to hang in. Then I did another:

<i>numbers</i>	<i>cubes</i>
1	1
2	8
3	27
4	64
5	125
15	225

Ummmm! Definitely interested. She took the pen and pad out of my hand and constructed a third one for herself:

<i>numbers</i>	<i>cubes</i>
1	1
2	8
3	27
6	36

Holy cow. Does it always work? “Yep, it sure does.” *That's neat!*

“Okay—as a reward for that lovely smile, you get blown away. Pick any number. Less than 100.” Her eyes wandered ‘round the room. “Not *too* big. But lots of factors.”

24!

“Now make a list of all the factors of 24. Starting with 1 and ending with 24.

She made the list—wrote them down in order of size too.

“Now beside each factor, put the number of factors that *it* has.”

What do you mean?

“Well, like 6—what are its factors? 1, 2, 3 and 6 itself. That's 4 factors. So we put a 4 beside the 6.”

She did 1, 2 and 3, but then paused at 4—*Does 2 get counted twice?*

“No just once.”

So she constructed the second column.

“Now write down the cubes of the list you've just made.”

She knew just what to do. She listed the cubes (using her calculator of course!) and then added both columns.

wow. WOW!

Boy! (Girl?)—being a math teacher sure is wonderful when you get a reaction like that.

Table for $n = 24$		
Divisors of 24	# of divisors	cubes of col 2
1	1	1
2	2	8
3	2	8
4	3	27
6	4	64
8	4	64
12	6	216
24	8	512
SUM:	30	900

Well, here's how I met this problem. Ever since my high school days I have known the formula for the sum of the first n natural numbers:

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

and of their squares:

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

But a few years passed before I beheld the remarkable formula for the sum of their *cubes*:

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2} \right)^2$$

Of course I immediately wrote this as:

$$1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + 3 \dots + n)^2$$

and recited it as a poem. And that's what I was whetting Jo's appetite with at the beginning. As I recall when I first saw this formula, I right away decided there was a pattern here, and that almost certainly we'd get more of the same, for example:

$$1^4 + 2^4 + 3^4 + \dots + n^4 = (1^2 + 2^2 + 3^2 \dots + n^2)^2$$

But alas this was not to be. For example $1^4 + 2^4$ equals 17 which is not even a square. So I pondered silently over the "sum of cubes" anomaly and tried to interpret it in a way that would allow me to understand "why" it held (Problem 1), but did nothing further with it.. And there the matter rested for many years.

And then in Spring 1996 at a meeting in Halifax, one of the speakers, Ed Barbeau of the University of Toronto, presented the divisor construction. The number Ed got from the class was 72 which for my purposes here is a better number to use.

What Ed told us is that for any number N at all, if we list the divisors of N and then make a second list of the number of divisors of each divisor, then that second list has the property that "the sum of cubes is the square of the sum."

I was of course bowled right over. What on earth is going on here? In countless years I had never seen another natural example of my poem, and now all of a sudden here was a whole battalion of such examples.

In fact, what do these new examples have to do with the old one? Is it really new, or is this just the old property dressed up to go out on the town? I could hardly wait to get a moment by myself.

*the sum of the cubes
is the square of the sum*

<u>Table for $N = 72$</u>		
Divisors of 72	# of di- visors	cubes of col 2
1	1	1
2	2	8
3	2	8
4	3	27
6	4	64
8	4	64
9	3	27
12	6	216
18	6	216
24	8	512
36	9	729
72	12	1728
SUM:	60	3600

Once away from the noise of the meeting, I sat down on a bench under some trees, and set down a bunch of “small” examples.

<u>Table for $N = 5$</u>		
Divisors of 5	# of divisors	cubes of col 2
1	1	1
5	2	8
SUM:	3	9

<u>Table for $N = 9$</u>		
Divisors of 9	# of divisors	cubes of col 2
1	1	1
3	2	8
9	3	27
SUM:	6	36

<u>Table for $N = 8$</u>		
Divisors of 8	# of divisors	cubes of col 2
1	1	1
2	2	8
4	3	27
8	4	64
SUM:	10	100

<u>Table for $N = 10$</u>		
Divisors of 10	# of divisors	cubes of col 2
1	1	1
2	2	8
5	2	8
10	4	64
SUM:	9	81

<u>Table for $N = 12$</u>		
Divisors of 12	# of divisors	cubes of col 2
1	1	1
2	2	8
3	2	8
4	3	27
6	4	64
12	6	216
SUM:	18	324

<u>Table for $N = 16$</u>		
Divisors of 16	# of divisors	cubes of col 2
1	1	1
2	2	8
4	3	27
8	4	64
16	5	125
SUM:	15	225

Well, there are certainly some patterns to be accounted for. For example, the $N=5$ table is small, having just a 1 and a 2. When will that happen?—*precisely when N is prime*. There’s one result.

But the striking observation belongs to the tables for $N = 8, 9$ and 16 . For these, column two is a sequence of the original kind (the integers from 1 to N) and so the “sum of cubes” property we have here is just the original one. Well now that’s *very* interesting. It suggests that we might be dealing here with a legitimate generalization of the original property.

But there’s more—Look! *Look at tables 8 and 9. What are the sums?—10 and 6. And their product is 60, which is the sum for 72. And 8 and 9 of course have product 72. Holy cow. There’s even some high level arithmetic structure.*

Okay, let’s slow down here. There’s something very nice happening and we don’t want to blow it. First of all, when will column two have the original form?—being exactly a set of consecutive integers? The answer is that this will happen N is a prime power. If $N = p^k$, for some prime p , then column one will consist of the powers of p from 0 to k (these are the divisors of p^k), and column two will therefore consist of the integers from 1 to $k+1$. So in that case we have the same property we had before, though a new interpretation of it.

The product law.

Now for the next observation. How are we to organize and understand the relationship between the 72-table and the 8- and 9-tables?

Let's start by asking how the divisors of 72 are related to the divisors of 8 and 9. Of course 72 is the product of 8 and 9, but more than this is true—this decomposition of 72 follows the prime factorization. That is, there are two primes in the factorization of 72, 2 and 3, and the 8 collects the 2's and the 9 collects the 3's:

$$72 = 8 \times 9 = 2^3 \times 3^2$$

What this means is that the divisors of 72 are exactly the products of the divisors of 8 and the divisors of 9. It's important to be careful here. Certainly if I take a divisor of 8 and a divisor of 9 and multiply them together, I'll get a divisor of 72, but I'm saying more than that. I'm saying that if we make a list of the divisors of 8 and of the divisors of 9, and then take all possible products of one list with the other, we'll get exactly the divisors of 72, with no repeats.

[To make this argument, we need some technical lore here about how the divisors of a number are obtained from the prime factorization. I reproduce the principle at the right.]

So now, let's ask about column two. To take an example, consider the divisor 12 of 72. I ask how many divisors 12 has, but I am going to try to find the answer, not by looking in column two in table 72 (where there's a 6 sitting right beside it) but by trekking over to tables 8 and 9. To do that, I write 12 as the product 4×3 so we look at the 4-row in the 8-table and the 3-row in the 9-table. Now column two in those two tables tells us how many divisors there are of each factor—4 has 3 divisors and 3 has 2 divisors. How many divisors does that give for 12?—well, $3 \times 2 = 6$ because the divisors of 4×3 are all the products of the divisors of 4 and the divisors of 3.

What has this told us?—that each column-two entry in the 72-table is the product of the corresponding column-two entries in the 8- and 9-tables. **YES!**

How to find the divisors of a number.

There's a wonderful general principle—if you happen to have the prime factorization of the number, then you have a procedure for writing down all the divisors. We illustrate this with the example of 2016? Its prime factorization is

$$2016 = 2^5 \times 63 = 2^5 \times 3^2 \times 7$$

so its divisors are all numbers of the form

$$2^a \times 3^b \times 7^c$$

where a can be anything between 0 and 5, b can be anything between 0 and 2, and c can be anything between 0 and 1? That gives me 6 choices for a , 3 choices for b , and 2 choices for c , for a total of

$$6 \times 3 \times 2 = 36 \text{ divisors.}$$

Table for $N = 8$		
Divisors of 8	# of divisors	cubes of col 2
1	1	1
2	2	8
4	3	27
8	4	64
SUM:	10	100

Table for $N = 9$		
Divisors of 9	# of divisors	cubes of col 2
1	1	1
3	2	8
9	3	27
SUM:	6	36

To emphasize this, I rewrite the 72-table replacing the entries by the appropriate products. This also prompts me to reorder the rows to make the structure easier to see.

This way of writing the 72-table shows in a very precise sense what it means to assert that this table is the “product” of the 8-table and the 9-table.

A more formal way to set this out, actually working with the equations, is found below, and in fact this form is probably the orthodox mathematical approach. It shows quite precisely that we can get the “sum of cubes = square of sum” result for column two of the 72-table, assuming only the familiar result for sets of the form $\{1, 2, 3, \dots, n\}$. The steps are technically interesting, and considerable care is required to get them in the right order.

Table for $n = 72$		
Divisors of 72	# of divisors	cubes of col 2
1=1·1	1=1·1	1=1·1
2=2·1	2=2·1	8=8·1
4=4·1	3=3·1	27=27·1
8=8·1	4=4·1	64=64·1
3=1·3	2=1·2	8=1·8
6=2·3	4=2·2	64=8·8
12=4·3	6=3·2	216=27·8
24=8·3	8=4·2	512=64·8
9=1·9	3=1·3	27=1·3
18=2·9	6=2·3	216=8·27
36=4·9	9=3·3	729=27·27
72=8·9	12=4·3	1728=64·27
SUM:	60=10·6	3600=100·36

Doing the proof with equations	
$1^3 + 2^3 + 3^3 + 4^3 + 2^3 + 4^3 + 6^3 + 8^3 + 3^3 + 6^3 + 9^3 + 12^3$	Here's the sum of cubes from the big table.
$= 1^3 1^3 + 2^3 1^3 + 3^3 1^3 + 4^3 1^3 + 1^3 2^3 + 2^3 2^3 + 3^3 2^3 + 4^3 2^3 + 1^3 3^3 + 2^3 3^3 + 3^3 3^3 + 4^3 3^3$	In order to move over to the two small tables, we have written each divisor as a product of its factors.
$= (1^3 + 2^3 + 3^3 + 4^3)(1^3 + 2^3 + 3^3)$	We factored the big expression, essentially separating out the two small tables.
$= (1 + 2 + 3 + 4)^2(1 + 2 + 3)^2$	We used the familiar “sum of cubes = square of sum” result in each small table.
$= [(1 + 2 + 3 + 4)(1 + 2 + 3)]^2$	We prepared the way to move back to the big table.
$= [1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 + 4 \cdot 1 + 1 \cdot 2 + 2 \cdot 2 + 3 \cdot 2 + 4 \cdot 2 + 1 \cdot 3 + 2 \cdot 3 + 3 \cdot 3 + 4 \cdot 3]^2$	We expanded the expression in order to get the entries from the big table.
$= (1 + 2 + 3 + 4 + 2 + 4 + 6 + 8 + 3 + 6 + 9 + 12)^2$	Now we have the square of the column 2 sum from the big table.

The calculations in this table are quite elegant, and it is a fine challenge for a student to set them down coherently. But for me anyway, the first approach which just looks at the relationship between the tables is already completely convincing and provides the best way to “see” what’s happening.

Problems

1.(a) Use the diagram at the right to find a geometric argument that

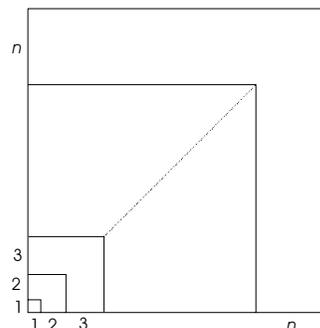
$$1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + 3 \dots + n)^2$$

(b) Modify the picture to enable you to get a formula for the sum of the odd cubes:

$$1^3 + 2^3 + 3^3 + \dots + n^3$$

in terms of the corresponding sum of the odd numbers:

$$1 + 2 + 3 + \dots + n.$$



2.(a) Suppose n , m , and N are integers and $N = mn$. If m has 4 divisors and n has 5 divisors, does it follow that N has 20 divisors? Give an example to show that the answer is NO.

(b) Can you put a condition on m and n to make the answer YES? If so, what is it?

3. As a mathematician, I do not stop with what we have discovered so far—I try to understand what’s going on at a *higher level of generality*. We have been talking about sets of numbers which have the property “sum of cubes = square of sum”. Maybe we need a name—let’s call such a set a CS-set (for Cube and Square). The original CS sets are the sets $\{1, 2, 3, \dots, n\}$, and that was our starting point. Now we have found a new class of CS-set, obtained by working with the divisors of a number N . And we ask right away whether there is any relationship between these new sets and the ones we already know about. And that’s what our investigation has been so far.

And we have discovered that these new CS-sets are in a precise sense “products” of the original CS-sets, and the factors in this product are obtained from the prime-power factorization of N —really quite a lovely result in that it displays an “isomorphism” between two types of arithmetic structures.

Let’s give a name to the sets which arise as column-two sets for some number N —we’ll call them *D-sets* (for “divisor”). For example, every set of the form $\{1, 2, 3, \dots, n\}$ is a D-set (for $N = p^{n-1}$ for any prime p). In this terminology, the main result of this section is that every D-set is a CS-set. Now here are some problems

(a) Are there CS sets that are not D-sets? Well, yes there are. Show that the sets $\{1\}$, $\{2,2\}$, $\{3,3,3\}$, $\{4,4,4,4\}$ etc. are all CS-sets. Why (except for the first) can they not be D-sets? Call these the C-sets (for “constant”).

(b) Is the product of any two T-sets again a T-set?

(c) Is the product of any two D-sets again a D-set?

(d) Show that the “product” of two CS-sets is again a CS-set. [Hint: the manipulations you require for this are essentially those that we used above to show that the product of the sets $\{1,2,3,4\}$ and $\{1,2,3\}$ is a CS-set, except here you have to do them “in general.” This simple result was “missed” by the mathematicians who first studied these sets.]

(e) What about the product of a T-set and a D-set? Is it another D-set or is it something new? Give a number of examples.

(f) A possible conjecture is that every CS-set is the product of a D-set and a T-set. Investigate this. [Screwball hint: Find integral solutions to $(x + y)^2 = x + y^3$.]