“Training” our students.

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Introduction

We focus principally on the high school mathematics curriculum. Our observation is that for most students it is lifeless and ineffective. Our thesis is that the curriculum should be implemented as a sequence of good examples designed to expose not content so much as “method.” This sounds like a simple enough idea but if fully implemented it is in fact quite radical and is steadfastly resisted by most of our mathematical colleagues.

What do we mean here by “good” examples? When we attempt to construct such examples, we are aware of being pulled in opposite directions, as if the example were a stretched spring bearing the tension between two forces, for example, the elementary and the sophisticated, the concrete and the abstract, the important and the whimsical. For all the apparent opposition of each of these pairs, as we penetrate the example we find that one so often seems to hold the key to the other. Thus the spring not only holds the tension, but also resolves it.

In addition we might trot out a list of fine adjectives like "lively" and "imaginative" but at the very least, a good example should be interesting to the teacher. This is a serious comment. It is alarming how many teachers take material into the classroom that is not of much interest to them. This attitude is inevitably conveyed to their students.

Following our own advice, we should begin with a good example.

Example—building trains.

We want to build a train out of cars. Now we have three kinds of cars, one kind of length 1, and two kinds of length 2, type A and type B. The question is how many different kinds of trains of total length 12 can we build? Note that a train is an ordered set of cars, with a front and a back. For example, there are 5 trains of length 3—111, 1A, A1, 1B, B1. A variation on this problem has been often used as a high school project. It supposes there is only one kind of car of length 2, and leads to the Fibonacci numbers.

This is a very rich exploratory problem but here we will present (rather directly) one possible route. Let t_n denote the number of trains of length *n*. We work out the first few terms of the t_n sequence simply by listing the possibilities and find that $t_1 = 1$, $t_2 = 3$, $t_3 = 5$, $t_4 = 11$, etc. What can we do? Look for a pattern in the numbers? We might need more of them to do that successfully.

An important general strategy is to try to find a recursive formula for the number of trains of length *n* in terms of the numbers of shorter trains. Here’s the argument. We partition the set of all *n*-trains into three disjoint classes distinguished by the type of the first car, the possibilities being 1, A and B. Then we count the number of trains of each type and add them up. Well if the first car is of type 1, the train gets completed with a train of length *n*–1, so there are t_{n-1} of these. *A*nd if the first car is of type A or B, the train gets completed with a train of length *n*–2, and there are t_{n-2} of each of these. This gives us the recursion

$$t_n = t_{n-1} + 2t_{n-2}.$$

The above sequence clearly conforms to this rule, and we can now continue it to find t_{12}:

1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, 2731.

We deduce that there are 2731 different trains of length 12.

Well we’ve answered the question but other questions immediately arise. What exactly is 2731? What is its relationship to 12? Is there a general formula for t_n?

When we first gave that question to a grade 12 class we were surprised at how quickly they produced the formulae.

$$t_n = \begin{cases} \dfrac{2^{n+1}+1}{3} & n \text{ even} \\ \dfrac{2^{n+1}-1}{3} & n \text{ odd} \end{cases}$$

How do they do it?—well, they use a variety of *ad hoc* and quite inventive methods. One approach is to take differences, not of successive terms, but of every *second* term. That is, looking at only the odd numbered terms: 1, 5, 21, 85, … we get the differences 4, 16, 64,… which are powers of 4, and looking at the even numbered terms: 3, 11, 43, 171,… we get the differences 8, 32, 128,… which are twice the same powers of 4. And from these they cobble the formulae. Another approach is to observe that the sequence is more or less doubling each time, so compare it with 2^n. When we tabulate them together (try it) we are struck that the powers of 2 are the sums of adjacent terms: 1+3=4, 3+5=8, etc. and one proceeds from here.

But there are in fact general "methods" for tackling such problems, and after this *ad hoc* phase, the time is ripe to look at one of these. We start by restating the problem. We want to "solve" the recursion

$$t_n = t_{n-1} + 2t_{n-2}$$

with the initial conditions, $t_1 = 1$ and $t_2 = 3$.

The key idea is to relax our requirements somewhat—to expand our net. Specifically we abandon our specification of t_2. So we want $t_1 = 1$ and we want the recursion to hold—but that's all. And we ask what solutions there might be. In particular we will be interested in "simple" solutions—solutions which are really easy to describe. *Okay, find some solutions!*

The students quickly see that if t_1 is required to be 1, then once t_2 is chosen, the rest of the sequence is specified. So all we have to do is to take different possibilities for t_2 . We get a bunch of solutions put up on the board. The third one is the original train sequence.

(a) 1, 1, 3, 5, 11, 21,...
(b) 1, 2, 4, 8, 16, 32,...
(c) 1, 3, 5, 11, 21, 43,…
(d) 1, 4, 6, 14, 26, 54,…
(e) 1, –1, 1, –1, 1, –1,...
(f) 1, –2, 0, –4, –4, –12,...
(g) 1, 1/2, 5/2, 7/2, 17/2, …

Fractions and negative numbers are perfectly permissible. Now the question is this—if you wanted to choose a solution which was easy to describe, what would it be?

Well, we get two answers—(b) and (e). In fact, both of these are power sequences, the powers of 2, and the powers of –1. We take these as "special" sequences and ask: *can we find a simple relationship between the original train sequence and these special sequences?*

And the answer is yes. That annoying business of a different formula for odd and even terms—is not the 1, –1 sequence exactly what's needed to accommodate that? The formula becomes:

$$t_n = \frac{4}{3}2^{n-1} - \frac{1}{3}(-1)^{n-1}.$$

A little fiddling is needed to get the constants right, but there's also a systematic approach. Once we've decided to look for a linear form: $t_n = au_n + bv_n$, we can simply plug in the first two terms of all three sequences and we have two equations in *a* and *b* which we can easily solve. The bottom line is that we have found an interesting method for finding the formula for *t*, a method that will work quite generally when it is not so easy to guess what the formula might be. First we free up an initial condition to expand our set of solutions, secondly we look for special solutions that are easy to describe, and thirdly we try to express the desired solution as an average of these special solutions.

Analysis of the example.
Our early appreciation of music is probably elemental, but as our musical environment grows and widens, we develop a more sophisticated taste. This sophistication can come without much in the way of analysis, and requires only careful, single-minded listening. But in order to understand and, harder still, articulate this sophistication, we need something in the way of analytical experience. And for this purpose, we always find that it is the most elementary passages, the simplest sequences of notes, that are the most powerful in conveying the essence of the structural sophistication.

The trains problem seems a good example of this principle. It contains a number of sophisticated structures, such as the idea of modelling the construction recursively, the idea of relaxing the initial conditions, the identification of certain “special” solutions (eigensolutions), the linearity of the solution space which allows us to express desired solution in terms of these special solutions and thereby obtain a formula for it. These are important ideas in mathematics but they are typically encountered for the first time only in university and then in more complex environments, such as matrix theory, and systems of differential equations. Just as Beethoven and Mozart should be experienced long before university, our point is that sophisticated mathematical ideas should be encountered and worked with at the school level, but in an elementary context.

"The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact". John Coleman is fond of quoting this famous sentence from Whitehead's *Science and the Modern World.* Of course it works both ways. First of all abstractions are apprehended through a series of concrete experiences, and secondly the abstractions themselves allow us to understand the particular instances in a way that affords us the opportunity to recognize them in a new situation and gives us a key to unlock its structure. The pleasure that comes from such experiences is certainly akin to that of hearing different forms of a simple refrain recurring in a symphony. The underlying ideas of linearity and basis and eigenvalue analysis are excellent examples of such powerful abstractions and the trains example provides a nice concrete experience that embodies it. When working with this example at the high school level there is certainly no need to make heavy weather of the abstract notions.

On the other hand we have introduced this example with senior mathematics majors at university, all of whom have encountered the abstractions in many contexts, and we typically find they have enormous difficulty recognizing them. Though they may have studied matrix algebra, differential equations and Fourier analysis, with rare exceptions, they have no idea what an eigenvector really is. Though they have formally diagonalized a symmetric matrix, they don’t in any sense recognize this as a fundamental “method” for solving equations. Thus, the abstract notions seem to have given them little help in finding a solution to the trains problem.

We suggest that this is due to a severe lack of experience with good concrete examples. One needs to be ready for abstraction. Whitehead himself pointed this out in no uncertain terms in his *Aims of Education* when he warned that the stage of precision (the second stage of learning) will be barren and the stage of generalization (the third stage of learning) will be meaningless if enough good time is not spent with the inventiveness and the wonder of the stage of romance (the first stage of learning). Our impression of many university math majors is that their mathematical universe is an austere world of abstract objects and theorems, with little elemental experience with the rocks and trees and trains.

How do we alert the students to the fact that there are big issues around here? How do we point to the significance of the eigensolutions without getting caught up in excessive abstraction or complexity? A strategy we have used with this example is to bring a double spring to class (two springs with weights on the end in series) and observe that this physical system exhibits the same type of behaviour as the trains problem. With a random pair of starting displacements for the weights the resulting motion is chaotic. But there are two special starting configurations from which the motion is very simple: both weights executing simple harmonic motion with the same frequency. To get the first of these we pull both weights down, the bottom one twice the displacement of the top, and the two weights will oscillate in phase with a low frequency. To get the second we pull the top weight down, and raise the bottom one half the amount and the two oscillate 180 degrees out of phase with a higher frequency. It turns out (assuming no damping) that with *any* starting condition, the general motion can always be described as a linear combination of these two simple motions. [And the two special solutions are of course the eigensolutions of the linear system.]

Let us observe that this possibility of a physical realization of a mathematical idea is hardly anomalous. Much of mathematics was inspired and developed from a study of physical systems.

A curriculum structure based in method

The curriculum structure we are trying to describe here can be said to be based, not in knowledge, but in method. “Method“ refers to the manner in which scientists or artists take apart the universe, the way in which they seek to understand their world; it governs their choices—what to think about and how to think about it; what to do and how to do it. It is more concerned with general strategies than with particular theorems. It distinguishes between what is fundamental and what is derivative.

A method-based curriculum has an openness which is important for those (student and teacher) who would participate in shaping their own learning. Because there is a clear opportunity to select interesting imaginative examples an examples-based curriculum offers a much stronger invitation to both students and teachers to take part in the decisions about what will be learned and how. This isn't only about trying to meet their interests, it is more fundamentally about providing more opportunity for collaborative activities in which students and teachers design certain essential aspects the curriculum together.

Make no mistake that this shift in structure has enormous implications for the way in which the curriculum unfolds in the classroom and the shape of the resource materials. This is not surprising when one realizes, for example, that that the skeleton has an enormous influence on the behaviour of the organism. The archeologists can tell us unbelievable things about the daily routine of an individual whose million-year old skeleton they have found intact.

In a curriculum structured on method, the examples are chosen with regard to the methods or ways of thought that they illuminate and the ordering of examples follows this lead, rather than being based on the development of the knowledge base. Of course method and knowledge are closely related and evolve in parallel, and it will be quite natural to provide a coherent development of knowledge at the same time. However, in a method-based curriculum, what we pay attention to is the quality, coherence, transparency and completeness of the *examples* not of the knowledge base that they might draw upon. As a consequence there will be components of knowledge which are omitted or which appear in only a restricted version. In our debates with colleagues they are quite worried about what they call “gaps” and when these arise they insist on filling them in. This is insulting to a method-based curriculum which requires an artistic restraint at just this point. We are reminded of the old adage that the mark of a good teacher is not what she says but what she is prepared to leave out.

There is a good analogy here with the use of a parable to convey a general truth. The teacher does not choose to give a parable because she thinks that the student is not clever enough to understand the abstract principle, it is rather because the parable in its everyday immediacy has a flexibility and an openness that the general principle lacks. For the same reason it is generally wrong to follow the parable with a formulation of the general principle. To realize the power of the parable, the student wants to incorporate it into his own scheme of knowledge in the right way at the right moment. It is the simplicity and concreteness of the parable that allows it to live in the student’s mind (or heart) until that moment arrives.

It would be wrong to jump to the conclusion that for us knowledge is somehow tainted and a curriculum based on knowledge is bound to cause the entire subject to spoil. In fact the mathematical knowledge we now possess is nothing short of awesome and there's no reason it could not be the basis of a wonderful curriculum. The problem is that in mathematics at least at the high school level (and perhaps the first year or so of university) a knowledge-based curriculum seems almost inevitably to degenerate into a list of propositions and technical skills and without an inventive teacher with superior resource materials, this squeezes the life out of the subject.

We find that colleagues in other disciplines often do not understand what the fuss in mathematics is all about. For them, the distinction between knowledge and method is not so problematic. Though their subject also has coherence and a base of technical skills, they do not seem so worried as do mathematicians about gaps in a student's knowledge. There appears to be not the same tendency in, say, psychology or biology or literature for a knowledge-based curriculum to degenerate into a list of skills or facts (though it can happen).

When we make this comparison between mathematics and other subjects, our colleagues immediately assert that mathematics really *is* different, that it has a particular logical and hierarchical structure that make gaps in knowledge much more problematical than in biology or English. We agree that mathematics is different, but we do not agree that it cannot easily sustain gaps in knowledge. Of course in every discipline, there are broad areas of study that must not be missed. In English, one must study Shakespeare and modern poetry though it does not matter so much which plays and poems are studied. In biology one must study mammals and genetics, though it does not matter which species or processes are studied. In mathematics one must study trigonometry and geometry, though it does not matter so much whether the formula for sin(A+B) and the cyclic quadrilateral have been studied. But this is precisely the point at which we get disagreement from most mathematicians.

Teaching the method-based curriculum.

It will not surprise you that we are particularly interested to see our approach used for the mathematical and scientific education of future school teachers at the undergraduate stage. A rather particular reason for this of course is that teachers typically teach the way they have learned. Therefore, if we want teachers to use an example-oriented "methodological" approach in their classroom, we have to teach them mathematics this way.

In this regard, let us end with a proposal that will at the very least give significant pause to our colleagues. First take hold of the ideal math text book for grades 11 and 12 (which will never quite be written but that's merely a practical detail) and use that to structure the undergraduate curriculum for these students. The idea here is to take each section in the book and wring (ring!) what mathematics one can (or will!) from it. Our claim is that when we have reached the end of the book, we should be able to confidently convocate the student. There will be gaps for sure but the students will have made a priceless discovery in terms of their own capacity to use their learning to expand their mathematical universe, and that is worth a denumerably infinite sequence of gaps. [This is not a new idea; we have experience with two successful courses of this type, one by Israel Halperin at Queen's and the other by Edwin Moise at Harvard, both in the '60's.]

Conclusion

Perhaps it comes down to life. Being alive means being sensitive to the challenges in our environment. It means responding to these challenges in a flexible and creative, rather than mechanized, way. When we teach mathematics in the conventional manner, as a linear, hierarchical list of algorithms, we give students the false impression that this is an acceptable way to live, at least in school. It is not. We live in a society that is constantly changing and innovating as we ourselves, teachers and students, change and grow. Mathematics is a discipline that is also constantly growing. As such, a curriculum should be a living object open to experience, change, innovation. An examples-based curriculum stands the best chance of being able to do so.

References

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