COHEN-MACAULAY AND SEMINORMAL AFFINE SEMIGROUPS IN DIMENSION THREE

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ABSTRACT. In this paper we study the Cohen-Macaulay property, depth, and seminormality of certain three dimensional affine semigroups S which are close to being normal. Our approach is to regard the semigroup ring $k[S]$ as a module over a monomial system of parameters, and is illustrated by many explicit examples.

1. INTRODUCTION

An affine semigroup S is a subset of $\mathbb{Z}^m$ containing 0, closed under addition, with a finite number of generators (i.e. there is a finite subset X of S such that every element of S is the sum of elements of X). Affine semigroups are also called affine monoids. In this paper we study the set $\mathscr{C}'$ of affine semigroups contained in $\mathbb{N}_d^3 := \{(x, y, z) \in \mathbb{N}^3 | x+y+z \equiv 0 \mod d\}$, generated by $(d, 0, 0), (0, d, 0), (0, 0, d)$, where d is a positive integer (which will normally be greater than or equal to 4) and some additional generators, with quotient group $G(S)$ equal to $\mathbb{Z}_d^3 := \{(x, y, z) \in \mathbb{Z}^3 | x+y+z \equiv 0 \mod d\}$. If $\mathbf{a} = (a_1, a_2, a_3) \in S$ then d divides $a_1 + a_2 + a_3$ so we can define $\deg \mathbf{a} = (a_1 + a_2 + a_3)/d$. If S is generated by elements of degree one then S will be called homogeneous. Any affine semigroup S has a finite minimal set of generators called $\mathrm{Hilb}(S)$ which, if S is homogeneous, will equal the set of all elements of degree one in S.

We are interested in relations between semigroup properties of S and algebraic properties of the semigroup ring $R = k[S]$ (k a field), in particular seminormality, depth, and the Cohen-Macaulay property. The ring R can be identified with the subring of $k[u, v, w]$ (u, v, w indeterminants) generated by monomials $\mathbf{x}^{\mathbf{a}} = u^{a_1}v^{a_2}w^{a_3}$ for $\mathbf{a} = (a_1, a_2, a_3) \in S$. The ring $k[S]$ is $\mathbb{N}$-graded by $\deg \mathbf{x}^{\mathbf{a}} = \deg \mathbf{a}$, and also has an $\mathbb{N}_d^3$ (or S) grading by $\deg \mathbf{x}^{\mathbf{a}} = \mathbf{a}$. Furthermore R has a natural system of parameters $f = u^d, g = v^d, h = w^d$. This (and powers or permutations thereof) is the only system of parameters consisting of elements homogeneous in the $\mathbb{N}_d^3$-grading (i.e. consisting of monomials). The depth

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of R is the length of the longest regular sequence in R consisting of homogeneous elements in the $\mathbb{N}$-grading. The ring R is Cohen-Macaulay (depth 3) if and only if every homogeneous system of parameters (in the $\mathbb{N}$-grading) is regular. In particular R is Cohen-Macaulay if and only if the sequence $\{f, g, h\}$ is regular. Regularity of the system of parameters $\{f, g, h\}$ is readily determined within the semigroup S. However we will see that the depth of R can be two, but no two of $\{f, g, h\}$ form a regular sequence. Things are not too bad, in that $\{f, g+h\}$ is always a regular sequence in R in the depth two case. Nonetheless depth two seems to be a bit further removed from semigroup properties of S than Cohen-Macaulay. It is convenient to define S to be Cohen-Macaulay if R is Cohen-Macaulay, S to have depth i if R has depth i, etc.

Our approach (section 3), which seems to be a novel and useful one, is to decompose R according to congruence classes mod d in S. This gives us a graphical way of understanding such questions as whether or not g is a zero-divisor mod f, etc. Then we illustrate this approach in section 4 by studying in detail the set $\mathscr{C}$ of affine semigroups S generated by $(d, 0, 0), (0, d, 0), (0, 0, d)$, all interior generators (a, b, c) with $a+b+c = d, a > 0, b > 0, c > 0$ and possibly some extra edge generators. This class of semigroups is large enough to contain many interesting possibilites, but small enough to permit us to characterize in an attractive way in terms of Hilb(S) those S that are Cohen-Macaulay or seminormal. Note that the affine semigroups $S \in \mathscr{C}'$ are positive, i.e. they have only one invertible element 0. The real cone $C(S)$ spanned by S is the positive orthant $\{(x, y, z) \in \mathbb{R}^3 | x \geq 0, y \geq 0, z \geq 0\}$. Note furthermore that S is simplicial, i.e. the cone $C(S)$ is spanned by rank $G(S)$ linearly independent vectors. This is important to us because only simplicial affine semigroups have a monomial system of parameters.

This paper grew out of earlier work [10] and [11] in dimension two (the homogeneous coordinate ring of a projective monomial curve) and [3], which focusses on seminormal affine semigroups in dimension greater than 2 which are not necessarily simplicial. A reference for questions about the relation between properties of R and S is [1] section 5. In particular Problem 10 of [1] asks for criteria for $K[S]$ to be Cohen-Macaulay in terms of Hilb(S), a difficult problem in general. Similarly, even though seminormality of an affine semigroup ring has a clear geometric characterization (Theorem 2.1), characterizing seminormality in terms of the Hilbert basis of S is not so immediate.

Throughout, $\mathbb{N} = \{0, 1, 2, \ldots\}$. Other notation introduced above, such as $\mathscr{C}', \mathscr{C}, R, f, g, h$ will also be used throughout the paper.

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Remark 1.1. We work in dimension three for two reasons. One is that we were curious about how some of our ideas in dimension two ([10], [11]) would work out in higher dimension. Enough interesting things happen in dimension three to keep us busy. Some of our ideas, for example the decompositon of R as the direct sum of ideals I_α, will work unchanged in higher dimension, leading to a description of S in terms of various types of arms. But we will lose the pictures, and also we expect that the depth will no longer depend just on the dimensions of the arms, but also on how they fit together. For example, if $B = k[e, f, g, h]$ and $I = (e, f) \cap (g, h)$ then (according to Macaulay 2 [5]) the depth of I as a B-module is 2, but if $J = (e, f) \cap (e, g)$ then the depth of J is 3. We expect the higher dimensional analogue of Theorem 4.8 to be hopelessly complicated. Also Theorem 2.2 fails in higher dimension.

2. Local Cohomology and Seminormality

Before we begin the main part of the paper we will review some properties of seminormality and local cohomology for affine semigroups. Seminormality is defined both for semigroups and rings, and has a long history in K-theory and algebraic geometry, going back to [12] and earlier. The semigroup S is seminormal if and only if $R = K[S]$ is seminormal. This is discussed, for example, in [9]. Seminormal semigroups have nice relationships with local cohomology and the Cohen-Macaulay property as discussed in [3]. Seminormality of S has a geometric characterization which suffices for our purposes.

Theorem 2.1. [9, Theorem 4.3] *Let S be an affine semigroup. Then S is seminormal if and only if $S = \cup[G(S \cap \tau) \cap \mathring{\tau}]$ where τ ranges over all faces of $C(S)$ and $\mathring{\tau}$ is the relative interior of τ.*

If $S \in \mathscr{C}'$ is seminormal then Theorem 2.1 says, for example with $\tau = C(S)$, that S contains all elements of $G(S) = \mathbb{Z}_d^3$ that lie in the interior of $\mathbb{N}_d^3$ (i.e. with all coordinates positive). This motivates the definition of the set $\mathscr{C}$ of affine semigroups.

It is immediate from Theorem 2.1 that if S is a seminormal affine semigroup contained in $\mathbb{N}_d^2$ and containing $\{(d, 0), (0, d)\}$ then $\mathrm{Hilb}(S)$ consists of evenly spaced points along the line segment between $(d, 0)$ and $(0, d)$. In particular S is normal. From this and [3, Proposition 5.1] it follows that

Theorem 2.2. *If $S \in \mathscr{C}'$ and S is seminormal then S is Cohen-Macaulay.*

Now we review some local cohomology, which we use to establish Theorem 2.5, and also to motivate the definition of S_0. Recall the following result from [2] (which is reviewed on page 372 of [3]).

Theorem 2.3. *Let $R = K[S]$ where S is any finitely generated commutative submonoid of $\mathbb{N}^m$ whose quotient group $G(S)$ is of rank d. Let $L^\bullet(S)$ be the complex*

$$0 \to L^0(S) \to \cdots \to L^t(S) \to \cdots \to L^d(S) \to 0$$

where $L^t(S) = \bigoplus R_F$, where the direct sum is taken over all faces F of $C(M)$ of dimension t and R_F is the homogeneous localization of R at the prime ideal of F. Then $H^i_{\mathfrak{m}}(R) = H^i(L^\bullet S)$ (as a $\mathbb{Z}^m$-graded R-module).

The definition of the homogeneous localizations R_F and of the boundary maps in $L^\bullet(S)$ are as in [3]. Furthermore recall the following facts about local cohomology (see [2] Theorem 3.5.7, at least for the local version):

Theorem 2.4.
1. *The depth of R is the smallest integer i such that $H^i_{\mathfrak{m}}(R) \neq 0$.*
2. *$H^d_{\mathfrak{m}}(R) \neq 0$.*
3. [3, Theorem 4.3] *If R (equivalently S) is seminormal and $H^i_{\mathfrak{m}}(R)_a \neq 0$ for some multidegree a then $a \in -C(S)$.*

Now let us examine the exact sequence of Theorem 2.3 if $S \in \mathscr{C}'$. Then the sequence is of the form

$$0 \longrightarrow L^0(S) \xrightarrow{\phi_0} L^1(S) \xrightarrow{\phi_1} L^2(S) \xrightarrow{\phi_2} L^3(S) \longrightarrow 0. \tag{1}$$

The prime ideal of each face F has K-basis the monomials $\mathbf{x}^{\mathbf{a}}$ where $\mathbf{a} \in S$, $\mathbf{a} \notin F$ and to form R_F we make invertible $\mathbf{x}^{\mathbf{a}}$ where $\mathbf{a} \in S \cap F$. There is only one zero dimensional face, namely the apex $\mathbf{0} = \{0,0,0\}$ of $C(S)$ and $\mathbf{x}^{\mathbf{0}} = 1$ is already invertible. Therefore $L^0(S) = R$. There are three one dimensional faces of $C(S)$, namely the coordinate axes, and $L^1(S) = R_f \oplus R_g \oplus R_h$. There are three two dimensional faces of $C(S)$, namely the coordinate planes, and $L^2(S) = R_{gh} \oplus R_{fh} \oplus R_{fg}$. There is only one three dimensional face, namely $C(S)$ so $L^3(S) = R_{fgh}$. The boundary maps ϕ_0, ϕ_1, ϕ_2 can be chosen so that they have matrices respectively $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$. The rings R_f, R_g, R_h and R_{fg}, R_{fh}, R_{gh} are all contained in R_{fgh}. The above matrices define an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\psi_0} \mathbb{Z}^3 \xrightarrow{\psi_1} \mathbb{Z}^3 \xrightarrow{\psi_2} \mathbb{Z} \longrightarrow 0. \tag{2}$$

Therefore the sequence (1) becomes exact after extension of scalars to R_{fgh}. Furthermore $\ker(\psi_1)$ has $\mathbb{Z}$-basis $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Therefore $\ker(\phi_1) = R_{fgh}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cap L^1(S) = \{(s,s,s) | s \in R_f,\ s \in R_g,\ s \in R_h\}$ which can be identified in an obvious way with $R_f \cap R_g \cap R_h = K[S_0]$, where S_0 is the semigroup $\{(a,b,c) \in \mathbb{N}^3 | (a,b,c) + \lambda(d,0,0) \in S,\ (a,b,c) + \mu(0,d,0) \in S,\ (a,b,c) + \nu(0,0,d) \in S\}$ for some $(\lambda,\mu,\nu) \in \mathbb{N}^3$.

This leads to

Theorem 2.5. $\operatorname{depth}(R) \geq 2$ *if and only if* $S = S_0$

Proof. By the above discussion $\ker \phi_1 = K[S_0]$ and $\operatorname{im}\phi_0 = K[S]$. By Theorem 2.3 $H^1_{\mathfrak{m}}(R) = \ker \phi_1 / \operatorname{im}\phi_0 = K[S_0 \backslash S]$ and by Theorem 2.4 $\operatorname{depth}(R) \geq 2$ if and only if $S = S_0$. □

Similarly one can define $S' = \{(a,b,c) \in \mathbb{N}^3 | (a,b,c) + \lambda(d,d,0) \in S,\ (a,b,c) + \mu(d,0,d) \in S,\ (a,b,c) + \nu(0,d,d) \in S\}$ for some $(\lambda,\mu,\nu) \in \mathbb{N}^3$. Then $K[S'] = R_{gh} \cap R_{fh} \cap R_{fg}$ and we know from [13] (using that S is simplicial) that R is Cohen-Macaulay (equivalently $\operatorname{depth}(R) = 3$, or $H^1_{\mathfrak{m}}(R) = H^2_{\mathfrak{m}}(R) = 0$) if and only if $S' = S$.

Remark 2.6. The definition of S' is slightly different from that given in [13], where S' is defined as $\{(a,b,c) \in \mathbb{N}^3 | (a,b,c) + \boldsymbol{\ell}_i \in S, 1 \leq \ell \leq 3\}$, where $\boldsymbol{\ell_i} \in U_i = \{(u_1,u_2,u_3) \in S | u_i = 0\}$. Our definition is easily shown to be equivalent to this. We wish to make it more explicit that S' can be computed within each congruence class $\mod d$.

The auxiliary semigroups S' and S_0 constructed above from S are still in $\mathscr{C}'$ and hence are $\mathbb{N}$ and $\mathbb{N}^3_d$-graded. However S' and S_0 (and the corresponding semigroup rings $K[S']$ and $K[S_0]$) need not be homogeneous. Furthermore f, g, h is still a homogeneous system of parameters for the rings $K[S], K[S']$ and $K[S_0]$.

3. The basis of S over W.

Let $S \in \mathscr{C}'$. Let W be the affine subsemigroup of S generated by $\{(d,0,0),(0,d,0),(0,0,d)\}$. Then $R = K[S]$ is an algebra over the polynomial ring $B = K[W] = K[f,g,h]$, where as usual $f = u^d, g = v^d$

and $h = w^d$. Furthermore R is a finitely generated B-module with a (unique finite) minimal monomial generating set $\mathbf{x}^T$, where $T \subset S$ (we regard the generating monomials as monomials in u, v, w). Explicitly let $\mathbf{h}_1 = \{d, 0, 0), \mathbf{h}_2 = (0, d, 0), \mathbf{h}_3 = (0, 0, d)$ (so that in the notation of the previous section $f = \mathbf{x}^{\mathbf{h}_1}$, similarly for g, h. Then $T = \{\mathbf{a} = (a, b, c) \in S | \mathbf{a} - \mathbf{h}_i \notin S, i = 1, 2, 3\}$. In [6] the following algorithm was described for computing T, starting from Hilb(S). Let $T_0 = \{(0, 0, 0)\}$ and recursively find $T_i = \{\mathbf{x} + \mathbf{y} | \mathbf{x} \in T_{i-1}, \mathbf{y} \in \text{Hilb}(S)\} \backslash T_i'$ where T_i' consists of all elements of the form $\{\mathbf{x} + \mathbf{y} | \mathbf{x} \in T_{i-1}, \mathbf{y} \in \text{Hilb}(S)\}$ which are brought into $\cup_{j<i} T_j$ by subtracting some sum of the $\mathbf{h}_i$. Eventually $T_\ell = \emptyset$ and $T = \cup_{i<\ell} T_i$. For convenience of notation we will refer to T as the "basis" of S over W, or simply as basis of S, (and $\mathbf{x}^T$ as the basis of R over B) even though $\mathbf{x}^T$ is in general not a basis for R as a B-module (in the sense of generating as the basis of a free module).

We will say that two elements $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ of $\mathbb{N}^3$ are in the same congruence class mod d if $\alpha_i - \beta_i$ is divisible by d, $1 \leq i \leq 3$. We are thinking mostly of $\boldsymbol{\alpha}, \boldsymbol{\beta}$ in $G(S)$, for which there are d^2 congruence classes $\mod d$, with representatives $\mathcal{C} = \{\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)\} | \alpha_1 + \alpha_2 + \alpha_3 \equiv 0 \mod d, 0 \leq \alpha_i \leq d - 1\}$.

We now have the following

Theorem 3.1. [6] *R is Cohen-Macaulay if and only if R is a free B-module, equivalently each congruence class mod d contains one basis element.*

Proof. That R is Cohen-Macaulay if and only if R is a free B-module is well known (see for example [4, section 2.5] for an exposition of this). There is at least one basis element in each congruence class mod d because $G(S) = \mathbb{Z}_d^3$. There are d^2 congruence classes mod d and the rank of R as a B-module is d^2, so R is a free B-module if and only if each congruence class contains one basis element. $\square$

Example 3.2. The set $\mathcal{C}$ defined above is a basis for $\mathbb{N}_d^3$ over W. Since $\mathcal{C}$ contains d^2 elements, $\mathbb{N}_d^3$ is Cohen-Macaulay (which is well known by a result of Hochster [2, Theorem 6.3.5]). Because $d-1+d-1+d-1 = 3d-3 < 3d$ elements of $\mathcal{C}$ can only be of degrees 0,1,or 2. Furthermore a and b can be assigned arbitrarily in the ranges $0 \leq a \leq d-1, 0 \leq b \leq d-1$, for if $a + b = 0$ we must have $c = 0$, if $0 < a + b < d$ we must have $c = d - a - b$ and (a, b, c) is of degree one, if $a + b = d$ we must have $c = 0$, and if $a + b > d$ we must have $c = 2d - a - b$ and (a, b, c) is of degree two.

We have programmed the above algorithm in Mathematica and used it to help formulate the results discussed below.

Suppose $\boldsymbol{\alpha} \in \mathcal{C}$. Let $S_{<\alpha>} = \{\mathbf{a} \in S | \mathbf{a} \equiv \boldsymbol{\alpha} \mod d\}$. Similarly define $G(S)_{<\alpha>}$. Congruence $\mod d$ induces a direct sum decomposition of $R = \oplus_{\alpha \in \mathcal{C}} R_{<\alpha>}$ where $R_{<\alpha>} = \oplus_{\mathbf{a} \in S_{<\alpha>}} R_{\mathbf{a}}$. It is clear that $R_{<\alpha>}$ is a B-module. Furthermore a minimal generating set for $R_{<\alpha>}$ as a B-module consists of those elements $\mathbf{x}^t \in \mathbf{x}^T$ for which $t \equiv \boldsymbol{\alpha} \mod d$.

Now suppose that S is a homogeneous element of $\mathscr{C}'$. For each $\boldsymbol{\alpha} \in \mathcal{C}$ let $T_{\boldsymbol{\alpha}} = \{\boldsymbol{\beta} \in T | \boldsymbol{\beta} \equiv \boldsymbol{\alpha} \mod d\}$. Let $\mathbf{b}_{\boldsymbol{\alpha}} = (b_1, b_2, b_3)$ be the inf of all the elements in $T_{\boldsymbol{\alpha}}$ (i.e. b_1 is the smallest first coordinate of any element in $T_{\boldsymbol{\alpha}}$, similarly for b_2 and b_3). If $T_{\boldsymbol{\alpha}}$ consists of only one element, then $\mathbf{b}_{\boldsymbol{\alpha}}$ is that element. Otherwise $\mathbf{b}_{\boldsymbol{\alpha}} \notin S$. It is convenient to subtract $\mathbf{b}_{\boldsymbol{\alpha}}$ from each element of $T_{\boldsymbol{\alpha}}$. This gives us a set of elements in W and the corresponding monomials in B generate a B-ideal $I_{\boldsymbol{\alpha}}$ which is isomorphic as a B-module (with a shift in grading) to $R_{<\alpha>}$. We will study the ideals $I_{\boldsymbol{\alpha}}$. The result can be translated back into S by adding $\mathbf{b}_{\boldsymbol{\alpha}}$ to d times the exponent of each monomial in $I_{\boldsymbol{\alpha}}$. (Monomials in $B = K[f, g, h]$ will be expressed as monomials in f, g, h. The exponents have to be multiplied by d to convert back to monomials in u, v, w). Similarly we have ideals $I'_{\boldsymbol{\alpha}}$ and $I^0_{\boldsymbol{\alpha}}$ corresponding to $K[S']$ and $K[S_0]$.

Theorem 3.3. *Let S be a homogeneous element of $\mathscr{C}'$ and for $\boldsymbol{\alpha} \in \mathcal{C}$ let $\mathbf{b}_{\boldsymbol{\alpha}}$ be as defined in the previous paragraph. Then $\mathbf{x}^{\mathbf{b}_{\alpha}}$ is a basis for $K[S']_{<\alpha>}$ ($\Leftrightarrow b_{\boldsymbol{\alpha}} \in S'$). Equivalently $I'_{\boldsymbol{\alpha}} = B$.*

Proof. Let $\mathbf{b} = (b_1, b_2, b_3) \in G(S)_{<\alpha>}$. If b_1 is less than the first coordinate of $\mathbf{b}_{\boldsymbol{\alpha}}$ then $\mathbf{b}$ cannot be brought into S by adding any positive multiple of $(0, d, d)$. Therefore $\mathbf{b} \notin S'_{\boldsymbol{\alpha}}$. Similarly for b_2 and b_3. Conversely $(\mathbf{b}_{\boldsymbol{\alpha}})_1$ is the first coordinate of some element of $T_{\boldsymbol{\alpha}}$. Therefore adding some positive multiple of $(0, d, d)$ brings $\mathbf{b}_{\boldsymbol{\alpha}}$ into S. Similarly for $(\mathbf{b}_{\boldsymbol{\alpha}})_2$ and $(\mathbf{b}_{\boldsymbol{\alpha}})_3$, so $\mathbf{b}_{\boldsymbol{\alpha}} \in S'$. ☐

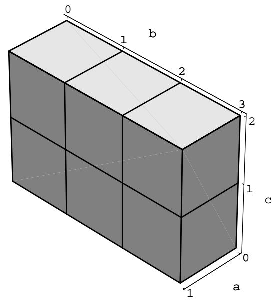
It follows from Theorem 3.3 that the B-basis of $K[S']$ contains d^2 elements, and hence that $K[S']$ is Cohen-Macaulay. Thus if $S = S'$ then S is Cohen-Macaulay. Conversely if $S_{\boldsymbol{\alpha}} \neq S'_{\boldsymbol{\alpha}}$ then S must have at least two basis elements in $T_{\boldsymbol{\alpha}}$ so S is not Cohen-Macaulay. This reproves the result from [13] mentioned after Therorem 2.5, that S is Cohen-Macalulay if and only if $S = S'$.

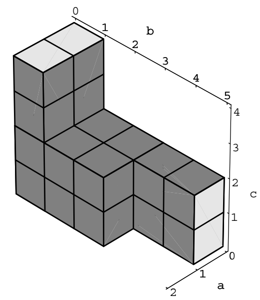
Theorem 3.4. *$I^0_{\boldsymbol{\alpha}}$ is the saturation of $I_{\boldsymbol{\alpha}}$ with respect to the homogeneous maximal ideal of B.*

Proof. It is clear from the definition of S_0 that a monomial $\mathbf{x}^{\mathbf{a}}$ is in $I^0_{\boldsymbol{\alpha}}$ if and only if $\mathbf{x}^{\mathbf{a}}$ multiplied by some power of each of f, g, h lies in $I_{\boldsymbol{\alpha}}$. ☐

The primary decomposition of $I_{\boldsymbol{\alpha}}$ can be computed by elementary combinatorial means (especially without the use of Groebner bases). For example we programmed [7, Theorems 5.23 and 5.27] in Mathematica. From the resulting primary decomposition one can easily describe those monomials not in $I_{\boldsymbol{\alpha}}$. The above cited theorems express $I_{\boldsymbol{\alpha}}$ as the irredundant intersection of irreducible (hence primary) monomial ideals. In the depth one case, for some $\boldsymbol{\alpha}$ there will be (at least) one irreducible ideal of the form $(f^a, g^b, h^c)B$ where a, b, c are all greater than zero. This irreducible monomial ideal will be represented by the triple (a, b, c), which we will call a box (in the language of Remark 1.1 this will be an arm of dimension 0). The saturation of $I_{\boldsymbol{\alpha}}$ is obtained by deleting these ideals from the primary decomposition. There may also be irreducible ideals in the decomposition where one of f, g, h does not occur. Missing parameters will be represented by 0. For example the ideal $(g^b, h^c)B$, $(b > 0, c > 0)$ will be represented by $(0, b, c)$, although as indicated below (∞, b, c) might be better. If $a > 0, b > 0, c > 0$ then monomials of the form $f^p g^q h^r$ where $0 \leq p < a, 0 \leq q < b, 0 \leq r < c$ are not in $I_{\boldsymbol{\alpha}}$. We can think of these as the monomials whose exponent lies in the box in $\mathbb{N}^3$ below (a, b, c). Similarly, for example, if $a = 0, b > 0, c > 0$ then monomials whose exponent lies in the infinite set consisting of all (p, q, r) where $p \geq 0$ is arbitrary and $0 \leq q < b, 0 \leq r < c$ are not in $I_{\boldsymbol{\alpha}}$. These exponents can be thought of as the box below (∞, b, c), which we will refer to as an arm, since it extends to infinity. (In the language of Remark 1.1 this will be an arm of dimension 1.) The monomials not in $I_{\boldsymbol{\alpha}}$ are exactly those of this form for at least one member of the primary decomposition. There can be no irreducible component with two of a, b, c equal to zero. For example, if $a > 0, b = 0, c = 0$ then adding no positive multiple of $(0, d, d)$ would bring $\mathbf{b}_{\boldsymbol{\alpha}}$ into S, contradicting the fact that $\mathbf{b}_{\boldsymbol{\alpha}} \in S'$.

Monomials in $K[S]$ that become zero divisors modulo one or more of the elements f, g, h can be computed in each $R_{<\boldsymbol{\alpha}>}$ and can be visualized using these descriptions of the monomials not in $I_{\boldsymbol{\alpha}}$. Suppose that for example that one of the ideals in the primary decomposition of $I_{\boldsymbol{\alpha}}$ is (f, g^3), which we represented by $(1, 3, 0)$. This is illustrated in the left hand diagram below, where the f, g, h axes are respectively labelled (a, b, c). Elements of $I_{\boldsymbol{\alpha}}$ are the points with integer coordinates on the dark surface and outside (as in the staircase diagrams of [7]). The light cross section indicates an arm that extends to infinity. Similarly the right hand diagram is a picture of the ideal $(f^2, g^3, h^2) \cap (f, h^2) \cap (f^2, g)$, which we represent by $\{(2, 3, 2), (1, 0, 2), (2, 1, 0)\}$. Here there are two arms, in the g-direction and the h-direction. These ideals actually arose as $I_{\boldsymbol{\alpha}}$ for some homogeneous $S \in \mathscr{S}'$ and some $\boldsymbol{\alpha}$.





In the present arguments it is convenient to identify monomials with their exponent vectors. Consider $I_{\boldsymbol{\alpha}}$ as in the left hand diagram. $I_{\boldsymbol{\alpha}}/fI_{\boldsymbol{\alpha}}$ can be visualized as the projection of $I_{\boldsymbol{\alpha}}$ along the f-axis. This will be all the points with integer coordinates on the left face of the dark surface except the corner (i.e. the points $(1, j, x)$ for $j = 0, 1, 2$ and x arbitrary) and all points on the g-h coordinate plane outside the arm (i.e. the points $(0, j, x)$ for $j \geq 3$ and x arbitrary.

Then for all integers $x \geq 0$, fh^x (corresponding to the line $\{(1, 0, x)\}$, $x \geq 0$ in the diagram) is (after multiplication by $\mathbf{x}^{b\alpha}$) an element of $K[S]_{<\boldsymbol{\alpha}>}$ which is non-zero in $(K[S]/fK[S])_{<\boldsymbol{\alpha}>}$ and is killed by g^3 but not by g^2. Similarly fh^xg represents a non-zero element of $(K[S]/fK[S])_{<\boldsymbol{\alpha}>}$ which is killed by g^2 but not g and fh^xg^2 represents a non-zero element of $(K[S]/fK[S])_{<\boldsymbol{\alpha}>}$ which is killed by g. In this case we see that $\{f, g\}$ is not a regular sequence in $K[S]$, even if the depth of $K[S]$ is two. However multiplication by h on $(K[S]/fK[S])_{<\boldsymbol{\alpha}>}$ just translates all points vertically, hence is injective. That is, $\{f, h\}$ is regular on $K[S]_{<\boldsymbol{\alpha}>}$, and will be regular on $K[S]$ if this is not ruled out by some other congruence class. Even if there are other irreducible components in $K[S]_{<\boldsymbol{\alpha}>}$ the above argument works for sufficiently large x. Similar arguments show that $\{g, h\}, \{h, f\}$, and $\{h, g\}$ are regular in $(K[S]/fK[S])_{<\boldsymbol{\alpha}>}$ but $\{g, f\}$ is not regular. More generally if $K[S]$ is of depth two there has to be an arm in some congruence class so a similar argument shows the following.

Theorem 3.5. *If $K[S]$ is of depth two then not all two element sequences from $\{f, g, h\}$ can be regular.*

Theorem 3.6. *If $K[S]$ is of depth two and the irreducible decompositions of the $I_{\boldsymbol{\alpha}}$ have arms in all three directions (either for the same or different $\boldsymbol{\alpha}$) then no two element subsequence of $\{f, g, h\}$ can be regular.*

Potentially one could have an indecomposable decomposition which has two components with a 0 in the same coordinate, e.g. $(2,1,0)$ and $(1,2,0)$. This can be thought of as two arms in the h-direction, or better yet, as a "stepped arm". The arguments that $\{f,g\}$ and $\{g,f\}$ are not regular in $K[S]_{<\alpha>}$, but $\{f,h\},\{g,h\},\{h,f\}$, and $\{h,g\}$ are remain the same. We have not found any stepped arms, but we have no theorems to rule them out (except for $S \in \mathscr{C}$).

Suppose that I_{α} has irreducible decomposition consisting of arms (of any shape) in two directions, say the f and h directions. Then the projection along the f-axis contains fewer points, but the behaviour of h on what is left is the same as above. Therefore $\{f,h\}$ is still a regular sequence on $K[S]_{<\alpha>}$. More generally we have

Theorem 3.7. *If $K[S]$ is of depth two and the irreducible decompositions of the I_{α} have arms in (at most) two directions (either for the same or different $\boldsymbol{\alpha}$), say the f and h directions. Then $\{f,h\}$ is a regular sequence in $K[S]$.*

Example 4.11 gives an explicit case of Theorem 3.6. Examples of both Theorems 3.6 and 3.7 exist in some abundance, as we will be seen in the last section.

If I_{α} is as in the right hand diagram above, with irreducible decomposition $\{\{2,3,2\},\{1,0,2\},\{2,1,0\}\}$, then the saturation of I_{α} (corresponding to $(S_0)_{<\alpha>}$ by Theorem 3.4) is represented by $\{\{1,0,2\}$, $\{2,1,0\}\}$ (i.e. just the arms remain). Note that in this congruence class $S_0\backslash S$ consists of four points corresponding to $(1,1,0),(1,2,0),(1,1,1)$, $(1,1,2))$. In fact it is clear that

Theorem 3.8. *For any $S \in \mathscr{C}'$, $S_0\backslash S$ is finite.*

Proof. The elements corresponding to $S_0\backslash S$ are some of the points inside a finite number of boxes. $\square$

Theorem 3.9. *If $K[S]$ is of depth two then $\{f, g+h\}$ is always a regular sequence.*

Proof. Since f and $g+h$ are both homogeneous of degree one, we can work in each congruence class. Furthermore $(R/fR)_{<\alpha>} \cong I_{\alpha}/fI_{\alpha}$ so we can work with I_{α}/fI_{α}, which we visualize as described above. If the class of e in I_{α}/fI_{α} is represented by a sum of monomials in $I_{\alpha}\backslash fI_{\alpha}$ then multiplication by the class of $g+h$ replaces each such monomial by a sum of two monomials, one shifted in the g-direction, and the other shifted in the h-direction. Because we have only arms not both of these shifted monomials can be zero in I_{α}/fI_{α} (i.e. "fall off an edge") and a monomial m of highest degree in g (on top of an arm in the g-direction) or highest degree in h (on top of an arm in the h-direction) (or if there are monomials in e on no arm, highest degree in either g or h) cannot have both of the monomials in $(g+h)m$ cancel off, proving that e cannot be killed by $g+h$. □

In contrast to the depth two case, the monomial in $I_{\boldsymbol{\alpha}}/fI_{\boldsymbol{\alpha}}$ corresponding to $(2,2,1)$ in the right hand diagram above would be killed by $g+h$.

4. Affine semigroups containing all interior generators.

In this section we consider the set $\mathscr{C}$ of affine submonoids S of $\mathbb{N}_d^3$, $d \geq 4$, generated by $\{(d,0,0),(0,d,0),(0,0,d)\}$, all elements of the form (a,b,c) with $a,b,c>0, a+b+c=d$, and some additional generators on the edges, i.e. of the form (a,b,c) with $a+b+c=d$ and one of a,b,c equal to 0. We expect to be able to say more about such semigroups than general ones. Note first of all that it is automatic for $d \geq 4$ that for $S \in \mathscr{C}$, $G(S)=\mathbb{N}_d^3$. Points in $\mathbb{N}_d^3$ with $a,b,c>0$ will be referred to as interior points and those with one of a,b,c equal to 0 will be called (interior) edge points. Interior points of degree one are automatically in $\mathrm{Hilb}(S)$ and will be referred to as $\mathrm{int}(\mathrm{Hilb}(S))$. The element of $\mathscr{C}$ with all edge generators is $\mathbb{N}_d^3$. If $S \in \mathscr{C}$ contains no points of degree one with 0 in the first coordinate except $(0,d,0)$ and $(0,0,d)$ then we will say that edge one (of $\mathrm{Hilb}(S)$) is empty, and if S contains all points of the form $(0,a,d-a)$ with $0 \leq a \leq d$) then we will say that edge one is full. Similarly for edges two and three.

Our goal is to determine which of the semigroups in $\mathscr{C}$ are seminormal and which are Cohen-Macaulay, keeping in mind that seminormality requires that all interior points be in S. The first step is to determine what basis elements of S over W can lie in the congruence class mod d of each element of $\mathfrak{C}$. Those elements of $\mathrm{Hilb}(S)$ with first coordinate 0 are the generators of the semigroup U_1 associated to a plane curve V_1 of degree d (or a divisor of d if $\{a_i|(0,a_i,d-a_i) \in \mathrm{Hilb}(S)\}$ has gcd greater than 1). If $\mathrm{Hilb}(S)$ contains no non-zero elements on this edge then U_1 is generated by $\{(0,d,0),(0,0,d)\}$ and S contains no non-zero basis elements with first coordinate 0. Otherwise U_1 will contain non-zero basis vectors over the semigroup generated by $\{(0,d,0),(0,0,d)\}$, and any such will be a basis element of S over W. Clearly there will be basis elements of U_1 in the congruence class mod d of $\boldsymbol{\alpha}=(0,a,d-a), 1 \leq a \leq d-1$ if and only if $\boldsymbol{\alpha} \in G(U_1)$. The corresponding results will hold if we similarly define U_i and V_i, $i=2,3$. In order simplify statements and proofs, assertions may be given only for one representative of each permutation class. For example, if a statement is made about $(d, 1, d-1)$ then it will be understood, without always being mentioned explicitly, that the corresponding results will hold for the 6 permutations of $(d, 1, d-1)$.

It is now immediate that

Theorem 4.1. [6] *If S is Cohen-Macaulay then so is U_i for $1 \leq i \leq 3$.*

Let $S \in \mathscr{C}$. It is convenient to define $\boldsymbol{\alpha} \in \mathcal{C}$ to be Cohen-Macaulay (for S) if $T_{\boldsymbol{\alpha}}$ contains only one basis element of S. (Recall that $T_{\boldsymbol{\alpha}}$ denotes the set of basis elements of S that are congruent to $\boldsymbol{\alpha}$ mod d). We have the following (and permutations thereof)

Theorem 4.2. *Let $S \in \mathscr{C}$ where $\mathscr{C}$ be the set of affine submonoids of $\mathbb{N}^3_d$, $d \geq 4$ defined at the beginning of this section and let $\boldsymbol{\alpha} \in \mathcal{C}$. Then*

(a) *If $\boldsymbol{\alpha} \in S$ then $\boldsymbol{\alpha}$ is Cohen-Macaulay.*

(b) *If $\boldsymbol{\alpha}$ is an interior element of $\mathcal{C}$ then $\boldsymbol{\alpha} \in S$, hence is Cohen-Macaulay.*

(c) *If $\boldsymbol{\alpha} = (0, a, d-a), 2 \leq a \leq d-2$ and $\boldsymbol{\alpha} \notin G(U_1)$ then $\boldsymbol{\alpha}$ is Cohen-Macaulay, and $T_{\boldsymbol{\alpha}} = \{(d, a, d-a)\}$.*

(d) *If $\boldsymbol{\alpha} = (0, a, d-a), 2 \leq a \leq d-2$ and $\boldsymbol{\alpha} \in G(U_1)\backslash S$ then $\boldsymbol{\alpha}$ is not Cohen-Macaulay. $T_{\boldsymbol{\alpha}}$ contains both $(d, a, d-a)$ and at least one basis element of U_1.*

(e) *If $\boldsymbol{\alpha} = (0, 1, d-1)$ then the only possible elements of $T_{\boldsymbol{\alpha}}$ in the interior of $C(S)$ are $(d, 1, d-1), (d, 1, 2d-1)$ and $(d, d+1, d-1)$.*

(f) *If $\boldsymbol{\alpha} = (0, 1, d-1) \notin G(U_1)$ and either (i) $(a, 0, d-a) \in \mathrm{Hilb}(S)$ with $2 \leq a \leq d-1$ or (ii) the only interior point on edge two in $\mathrm{Hilb}(S)$ is $(1, 0, d-1)$, and $(d-1, 1, 0) \in \mathrm{Hilb}(S)$, then $\boldsymbol{\alpha}$ is Cohen-Macaulay with $T_{\boldsymbol{\alpha}} = \{(d, 1, d-1)\}$. If (iii) the only interior point on edge two in $\mathrm{Hilb}(S)$ is $(1, 0, d-1)$, and $(d-1, 1, 0) \notin \mathrm{Hilb}(S)$, then $\boldsymbol{\alpha}$ is not Cohen-Macaulay with $T_{\boldsymbol{\alpha}} = \{(d, 1, 2d-1), (d, d+1, d-1)\}$. If (iv) edge two is empty then $\boldsymbol{\alpha}$ is Cohen-Macaulay with $T_{\boldsymbol{\alpha}} = \{(d, d+1, d-1)\}$. In no case can $T_{\boldsymbol{\alpha}} = \{(d, 1, 2d-1)\}$*

(g) *If $\boldsymbol{\alpha} = (0, 1, d-1) \in G(U_1)\backslash S$ and edge two is not empty then $\boldsymbol{\alpha}$ is not Cohen-Macaulay. $T_{\boldsymbol{\alpha}}$ contains some basis element of U_1 and either $(d, 1, d-1)$ or $(d, 1, 2d-1)$, as listed in (f)(i-ii or iii respectively). If edge two is empty then $\boldsymbol{\alpha}$ is Cohen-Macaulay if and only if $(0, d+1, d-1) \in S$. If $(0, d+1, d-1) \in S$ then $T_{\boldsymbol{\alpha}} = \{(0, d+1, d-1)\}$. If $(0, d+1, d-1) \notin S$ then $T_{\boldsymbol{\alpha}}$ contains $(d, d+1, d-1)$ and basis elements of U_1.*

Proof. Part (a) is clear. Let $\boldsymbol{\alpha} = (a, b, c)$. We say that $\boldsymbol{\alpha}$ is an interior element of $\mathcal{C}$ if $a, b, c > 0$. We saw above that the degree of $\boldsymbol{\alpha}$ can be at most two. If $\deg(\boldsymbol{\alpha}) = 1$ then $\boldsymbol{\alpha} \in \mathrm{int}(\mathrm{Hilb}(S))$. Since $\mathrm{int}(\mathrm{Hilb}(S)) \subset S$, $\boldsymbol{\alpha}$ is Cohen-Macaulay by (a). Suppose that the degree of $\boldsymbol{\alpha}$ is two. Since $1 + d - 1 + d - 1 = 2d - 1 < 2d$ we must have $a, b, c \geq 2$. It is then possible in many ways to write $\boldsymbol{\alpha}$ as the sum of two elements of $\mathrm{int}(\mathrm{Hilb}(S))$, so $\boldsymbol{\alpha} \in S$, again proving (b).

If $\boldsymbol{\alpha} = (0, a, d-a), 2 \leq a \leq d-2$ and $\boldsymbol{\alpha} \notin G(U_1)$ then $T_{\boldsymbol{\alpha}}$ contains no basis elements with first coordinate 0. Since $(d, a, d-a)$ can be written as the sum of two elements of $\mathrm{int}(\mathrm{Hilb}(S))$, we have $(d, a, d-a) \in S$. Every element of S that is congruent mod d to $(0, a, d-a)$ must lie in $W + (d, a, d-a)$ so $T_{\boldsymbol{\alpha}} = \{(d, a, d-a)\}$, proving (c). If $\boldsymbol{\alpha} = (0, a, d-a), 2 \leq a \leq d-2$ and $\boldsymbol{\alpha} \in G(U_1) \backslash S$ then $T_{\boldsymbol{\alpha}}$ must contain a basis element of U_1. But by assumption $(0, a, d-a) \notin S$, so $(d, a, d-a) \in T_{\boldsymbol{\alpha}}$, proving (d).

Now suppose that $\boldsymbol{\alpha} = (0, 1, d-1)$, which we assume not to be in S. We first ask if S can contain any elements of the form $(id, 1, jd-1)$ with $i \geq 1, j \geq 1$. If $(id, 1, jd-1) \in S$ then in the expression for $(id, 1, jd-1)$ as the sum of elements of $\mathrm{Hilb}(S)$ one element will contain the 1 in coordinate 2, the rest will be on edge two. If $\mathrm{Hilb}(S)$ contains an element of the form $(a, 0, d-a)$ with $2 \leq a \leq d-1$ then $(d, 1, d-1) = (a, 0, d-a) + (d-a, 1, a-1)$. But $(d-a, 1, a-1) \in \mathrm{int}(\mathrm{Hilb}(S))$ so $(d, 1, d-1) \in S$. If $\mathrm{Hilb}(S)$ contains only the point $(1, 0, d-1)$ in the interior of edge two, and also $(d-1, 1, 0) \in \mathrm{Hilb}(S)$ then $(d, 1, d-1) = (1, 0, d-1) + (d-1, 1, 0) \in S$. If $\mathrm{Hilb}(S)$ contains only the point $(1, 0, d-1)$ in the interior of edge two but $(d-1, 1, 0) \notin \mathrm{Hilb}(S)$ then $(d, 1, d-1) \notin S$, but $(d-1, 1, 2d-1) = 2(1, 0, d-1) + (d-2, 1, 1) \in S$ because $(d-2, 1, 1) \in \mathrm{int}(\mathrm{Hilb}(S))$. If $\mathrm{Hilb}(S)$ contains only the point $(1, 0, d-1)$ in the interior of edge two but $(d-1, 1, 0) \notin \mathrm{Hilb}(S)$ then no element of the form $(id, 1, d-1)$ with $i \geq 2$ can be in S. For if $(id, 1, d-1) \in S$ then in the expression for $(id, 1, d-1)$ as a sum of elements of $\mathrm{Hilb}(S)$ we must have $(id, 1, d-1) = a(d, 0, 0) + b(1, 0, d-1) + \boldsymbol{\gamma}$ where $\boldsymbol{\gamma}$ is an element of $\mathrm{Hilb}(S)$ with 1 in the second coordinate. We cannot have $b = 0$ because that would force $\boldsymbol{\gamma}$ to equal $(0, 1, d-1)$ which we assume is not in S. The only alternative is $b = 1$ which would force $\boldsymbol{\gamma}$ to equal $(d-1, 1, 0)$, which we also assume not to be in S. If edge two is empty and $(id, 1, jd-1) \in S$ with $i \geq 1, j \geq 1$ then we must have $(id, 1, jd-1) = a(d, 0, 0) + b(0, 0, d) + \boldsymbol{\gamma}$ where $\boldsymbol{\gamma}$ is an element of $\mathrm{Hilb}(S)$ with 1 in the second coordinate. This is possible only if $a = i$, which forces $\boldsymbol{\gamma}$ to equal $\boldsymbol{\alpha}$, which is assumed not to be in S. Therefore if edge two is empty then S contains no element of the form $(id, 1, jd-1) \in S$ with $i \geq 1, j \geq 1$. Finally note that $(d, d+1, d-1)$ can be written as the sum of three elements of $\mathrm{int}(\mathrm{Hilb}(S))$ so always $(d, d+1, d-1) \in S$. Altogether the only possible interior elements of

T_α are $(d,1,d-1),(d,1,2d-1)$ and $(d,d+1,d-1)$, which is (e). Cases (f) and (g) are also now clear. □

Corollary 4.3. *If $S \in \mathscr{S}$ then the only interior points of $\mathbb{N}_d^3$ that might not be in S are points of the form $(id,1,jd-1)$ for $i,j \geq 1$. In particular, S contains all interior points of $\mathfrak{C}$, which are then automatically basis elements of S.*

Proof. This is immediated from Theorem 4.2(f)(iv) and the corresponding part of (g). □

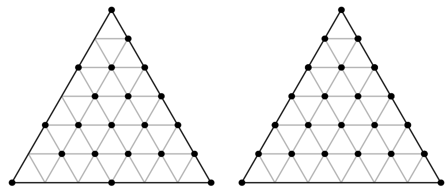
We are now ready to classify the seminormal elements in $\mathscr{C}$.

Theorem 4.4. *Suppose $S \in \mathscr{C}$.*

(a) *If S is seminormal then the edge elements of* Hilb(S) *are of the form $(ie, d-ie, 0)$ (and permutations thereof) where $e|d$, $1 \leq e \leq d$ and $0 \leq i \leq (d/e)$.*
(b) *If all the edge points of* Hilb(S) *are of the form described in (a), with $e < d$, then S is seminormal.*
(c) *The only other case where S is seminormal is where one edge is empty, and the other two edges are full.*

Proof. Note that the edges of Hilb(S), as described in (a), consist of evenly spaced points. If $e < d$ then Hilb(S) contains interior points of the edge. If $e = d$ then the only elements of Hilb(S) on the edge are $(d,0,0)$ and $(0,d,0)$. That the edges must be of this form if S is seminormal is immediate from Theorem 2.1. Conversely, again by Theorem 2.1, S will be seminormal if the edges are all of this form and S contains all interior points of $\mathbb{N}_d^3$. By Theorem 4.2(b)(d)(f)(g) the only points in the interior of $\mathbb{N}_d^3$ that might not be in S are points of the form $(id,1,jd-1)$ with $i \geq 1, j \geq 1$. But if each edge is of the form in (a) with $e < d$ then all points of the form $(id,1,jd-1)$ with $i \geq 1, j \geq 1$ are in S by Theorem 4.2(f)(i). If one edge (say the second) is empty then the only way that $(id,1,jd-1)$ can be in S is to have $(0,1,d-1) \in S$. In view of the restrictions in (a) this means that edge one is full. Similarly edge three is full, so (c) holds. □

The Hilbert basis of a homogeneous element of $\mathscr{S}'$ can be represented by a graph in barycentric coordinates, with $(d,0,0)$ at the top, $(0,d,0)$ at the lower left corner, and $(0,0,d)$ at the lower rignt corner. For example the following two graphs show Hilb(S) for typical examples of Theorem 4.4, part (b) on the left and part (c) on the right.



It is now easy to count the seminormal elements of $\mathscr{C}$.

Theorem 4.5. *Suppose that $d \geq 4$ has r positive divisors (including 1 and d). Then $\mathscr{C}$ contains $(r-1)^3 + 3$ seminormal semigroups.*

Proof. For the cases of Theorem 4.4(b) there are $r-1$ choices for each of three edges, together with the three possibilites from Theorem 4.4(c). $\square$

Note that if d is prime then $r = 2$ so there are only four seminormal elements of $\mathscr{C}$, $\mathbb{N}_d^3$ (in Theorem 4.4(b) all edges must be full) and the three from Theorem 4.4(c). If $d = 6$ then $r = 4$ so there are $27+3 = 30$ seminormal elements of $\mathscr{C}$. One should also remember that elements of $\mathscr{C}$ that differ by a permutation are counted separately.

Our next goal is to classify the Cohen-Macaulay elements of $\mathscr{C}$. Recall that S is Cohen-Macaulay if and only if each congruence class mod d contains only one basis element. Our first observation is that the possible edges of a Cohen-Macaulay S are somewhat restricted.

Theorem 4.6. *If S is Cohen-Macaulay then* $\mathrm{Hilb}(S)$ *on the edges can only be of the following form (illustrated for edge one).*

(a) *Evenly spaced points along the edge of the form $\{(0, \lambda a, d - \lambda a\} | d = qa, q \in \mathbb{N}, 0 \leq \lambda \leq q\}$. If d is prime the only two cases are $a = d$ in which case the edge is empty, or $a = 1$ in which case the edge is full.*

(b) *All edge points $\{(0, a, d-a) | 0 \leq 1 \leq d-1\}$ are included except for one or both of $a = 1, a = d-1$.*

Proof. By Theorem 4.2(d), if S is Cohen-Macaulay then $\mathrm{Hilb}(S)$ must contain all elements of the form $(0, a, d-a)$ with $2 \leq a \leq d-2$. This restricts $\mathrm{Hilb}(U_1)$ to the cases stated. $\square$

Due to the restriction in Theorem 4.2 the characterization of Cohen-Macaulay points simplifies to

Lemma 4.7. *Let $B = (0, 1, d-1)$. Suppose that $S \in \mathscr{C}$ with the edges of* $\mathrm{Hilb}(S)$ *as permitted by Theorem 4.6.*

(a) *If $B \in G(U_1) \setminus S$ then B is Cohen-Macaulay if and only if edge 2 is empty.*

(b) *If* $B \notin G(U_1)$ *then* B *is Cohen-Macaulay.*

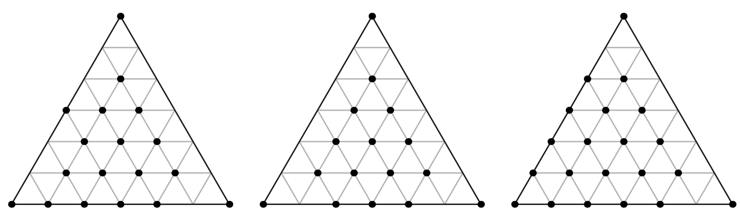
Proof. Part (a) occurs only for edges of the type in 4.6(b). In this case $(0, d+1, d-1) \in S$ so (a) follows from Theorem 4.2(g). Part (b) follows from Theorem 4.2(f) because case (f)(iii) cannot occur under the restrictions of Theorem 4.6. □

From Lemma 4.7 the following classification is immediate.

Theorem 4.8. *A semigroup* $S \in \mathscr{C}$ *is Cohen-Macaulay if and only if* Hilb(S) *is of the following form (or permutations thereof).*

(a) *The edges are either empty or proper seminormal, in any permutation.*
(b) *Edge one contains all points except* $(0, 1, d-1)$*, the point adjacent to the right end, edge three is empty or proper seminormal, and edge two is empty.*
(c) *Edge one in addition to the corners contains* $\{(0, a, d-a) | 2 \leq a \leq d-2\}$*, and edges two and three are empty.*
(d) *Edges one and three contain all points except* $(0, 1, d-1)$ *and* $(d-1, 1, 0)$ *respectively, and edge two is empty.*

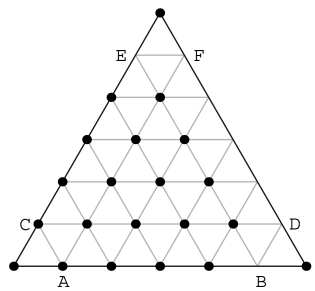
By "proper seminormal" we mean evenly spaced points along the edge, as in Theorem 4.6(a), with $a < d$. The Hilbert basis for examples of types (b), (c), (d) are shown below.



Corollary 4.9. *Suppose that* d *has* r *positive divisors. Then if* $d \geq 5$ *there are* r^3 *Cohen-Macaulay semigroups in* $\mathscr{C}$ *of type (a),* $6r$ *of type (b),* 3 *of type (c), and* 3 *of type (d) making a total of* $r^3 + 6r + 6$*. If* $d = 4$ *then case (c) is already included in case (a), making* $r^3 + 6r + 3 = 27 + 18 + 3 = 48$*, as observed in section 5.*

Proof. For case (a) there are r choices for each of three edges. For case (b) there are r choices for edge three, and 6 permutations. For cases (c) and (d) there are three permutations. □

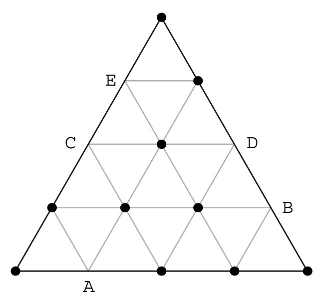
Below is an example of how Theorem 4.8 follows from Lemma 4.7.



Example 4.10. In the diagram above, which is an example of Theorem 4.8(d), points B and E are Cohen-Macaulay by Theorem 4.7(a), points D and F are Cohen-Macaulay by Theorem 4.7(b), and A and C are in Hilb(S) so are automatically Cohen-Macaulay.

The following example illustrates the ideals I_α and also Theorem 3.6.

Example 4.11. Consider the affine semigroup S with Hilbert basis illustrated below



Interior points of $\mathcal{C}$ are the only basis element in their congruence class, so it suffices to consider the missing points A $= (0,3,1)$, B $= (1,0,3)$, C $= (2,2,0)$, D $= (2,0,2)$, and E $= (3,1,0)$. The basis elements in the congruence classes of A, B, C, D, E respectively are $\{(0,3,5),(4,3,1)\}$, $\{((9,0,3),(1,4,3)\}$, $\{(2,6,0),(2,2,4)\}$, $\{(6,0,2),$ $(2,4,2)\}$, $\{(3,1,4),(3,9,0)\}$. These were found using our computer programs, but can be easily to checked using the discussion of Theorem 4.2. Note that the bases at B and E differ by a cyclic permutation, as do the bases at C and D, even though the diagram is not quite symmetric under rotation. So there are only three types of cases, those of A, B, C, for example. Then $b_{\mathrm{A}} = (0,3,1)$, subtracting b_{A} from each of the basis elements we get $\{(0,0,4),(4,0,0)\}$, and dividing by 4 we get $\{(0,0,1),(1,0,0)\}$, which are the exponents of the generators of I_{A}. This corresponds to indecomposable ideal $(1,0,1)$ which is an arm in the b-direction. Similarly B gives an arm $(2,1,0)$ in the c-direction and C gives an arm $(0,1,1)$ in the a-direction, and by cyclic permutation, D gives an arm $(1,1,0)$ in the c-direction and E gives an arm $(0,2,1)$ in the a-direction. Since we have only arms and no boxes, S is of depth 2. Since there are arms in all three directions no two of f, g, h is a regular sequence by Theorem 3.6. We obtain similar examples (with the same basis elements in the congruence classes of the missing points) if we include 0, 2, or 3 of the edge midpoints in Hilb(S), yielding after symmetries 16 affine semigroups C with $d = 4$, arms in three directions, and no boxes.

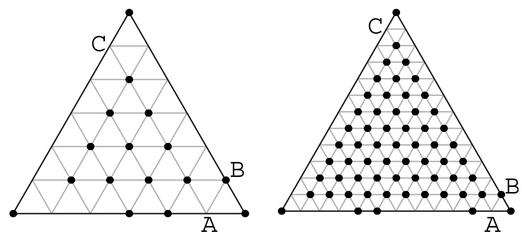
In degree 4 there cannot be three arms in one congruence class, but there can be in degree 5. In higher degrees there can be many basis elements of U_1 in the the congruence class of $(0, 1, d-1)$, which makes it possible to have complicated configurations of arms and boxes, as explained by the following discussion.

Remark 4.12. Let $\boldsymbol{\alpha} = (0, 1, d-1)$.

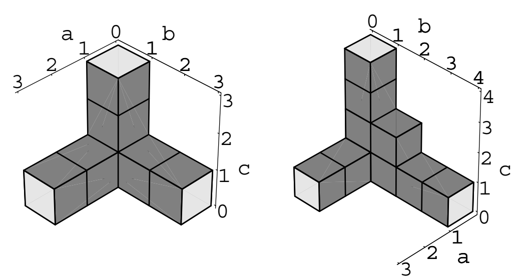
(a) Suppose that $T_{\boldsymbol{\alpha}}$ consists of $(d, 1, d-1)$ and basis elements $(0, 1+a_i d, (b_i+1)d-1), 1 \leq i \leq r$ from U_1 with $r \geq 1$, $a_i, b_i \geq 0$. Then $\mathbf{b}_{\boldsymbol{\alpha}} = (0, 1, d-1)$. Subtracting $(0, 1, d-1)$ and dividing by d we obtain $(1, 0, 0)$ and $\{(0, a_i, b_i)\}$ $(1 \leq i \leq r)$ as the exponent vectors of the generators of $I_{\boldsymbol{\alpha}}$. Since we started with a basis for $T_{\boldsymbol{\alpha}}$ we may assume $a_i < a_{i+1}$ for $1 \leq i \leq r-1$, which will imply $b_i > b_{i+1}$ for $1 \leq i \leq r-1$. If $a_1 > 0$ and $b_r > 0$ then by inspection the irredundant decomposition of $I_{\boldsymbol{\alpha}}$ is represented by $\{\{1, a_1, 0\}, \{1, a_2, b_1\}, \ldots \{1, a_n, b_{r-1}\}, \{1, 0, b_r\}\}$, which has arms in directions two and three, and $r-1$ boxes. If $a_1 = 0$ then y^{a_1} in the first element of the decomposition becomes $y^0 = 1$ and the first ideal in the decomposition becomes the unit ideal (redundant!) and the arm in direction three disappears. Similarly if $b_r = 0$ the arm in direction two disappears.

(b) Suppose that $T_{\boldsymbol{\alpha}}$ consists of $(d, d+1, d-1)$, $(d, 1, 2d-1)$ and basis elements $(0, 1+a_i d)$, $(b_i+1)d-1)$, $1 \leq i \leq r$ from U_1 with $a_i, b_i \geq 0$. Then again $\mathbf{b}_{\boldsymbol{\alpha}} = (0, 1, d-1)$. Subtracting $(0, 1, d-1)$ and dividing by d we obtain $(1, 1, 0)$, $(1, 0, 1)$ and $\{(0, a_i, b_i)\}$ $(1 \leq i \leq r)$ as the exponent vectors of the generators of $I_{\boldsymbol{\alpha}}$. Again we may assume $a_i < a_{i+1}$ and $b_i > b_{i+1}$ for $1 \leq i \leq r-1$. If $a_1 > 0$ and $b_r > 0$ then by inspection the irredundant decomposition of $I_{\boldsymbol{\alpha}}$ is represented by $\{\{0, 1, 1\}, \{1, a_1, 0\}, \{1, a_2, b_1\}, \ldots \{1, a_n, b_{r-1}\}, \{1, 0, b_r\}\}$, which has arms in all three directions, and $r-1$ boxes. If $a_1 = 0$ or $b_1 = 0$ then again the corresponding arm diasppears.

Example 4.13. An explicit example of Remark 4.12(b) is illustrated by the left hand graph below, with $\boldsymbol{\alpha}$ =A= $(0, 1, 5)$.



Here $T_{\boldsymbol{\alpha}} = \{((0,7,11),(6,1,11),(6,7,5)\}$. This is Remark 4.12(b) with $r = 1$, $a_1 = b_1 = 1$, so there are three arms and no box in the class of $\boldsymbol{\alpha}$. According to Theorem 4.2(g) the point B $= (1,0,5)$ must be the only point on edge two in Hilb(S) and C $= (5,0,1)$ cannot be in Hilb(S) in order for the example to work. The remaining four points on edge 3 might or might not be in Hilb(S), yielding 16 such cases, or $16 \times 6 = 96$ after permutations. None of these examples contains a box in any congruence class so all are of depth 2. Similarly the diagram on the right above, with $\boldsymbol{\alpha}$ =A= $(0,1,11)$ has $T_{\boldsymbol{\alpha}} = \{((0,13,35),(0,25,23),(12,1,23),(12,13,11)\}$ which is Remark 4.12(b) with $r = 2$, $a_1 = 1, b_1 = 2, a_2 = 2, b_2 = 1$, so there are three arms and one box in the class of $\boldsymbol{\alpha}$. The graphs of $I_{\boldsymbol{\alpha}}$ are illustrated below for these two cases.



Remark 4.14. There are many Cohen-Macaulay curves not of the type in Theorem 4.6, as discussed in various papers such as [8] or [10]. The number of such grows exponentially in d, even though the fraction of all curves that are Cohen-Macaulay is asymptotically 0 as shown in [10]. This is in contrast to the cases in Theorem 4.6 which grow only (at most) linearly in d.

5. Computational Observations

In this section we present the results of several computational surveys with $\mathscr{C}$. In degree d there are $d-1$ points along each edge which might or might not belong to Hilb(S), so that $\mathscr{S}$ contains $2^{3(d-1)}$ elements. For $d = 4,5,6$ respectively $\mathscr{C}$ will thus contain $2^9 = 512, 2^{12} = 4096$

and $2^{15} = 32768$ elements. There will be a certain amount of repetition due to symmetry, which we ignore in the first surveys. The method of computation is for each example to compute a basis of S over W. If there are d^2 basis elements the monoid is of depth three (Cohen-Macaulay). If there are more than d^2 basis elements but no boxes in the primary decomposition of any I_α (i.e. no irreducible component represented by (a, b, c) with $a, b, c > 0$) then S is of depth two, and if there is a box in some primary decomposition, then the depth of S is one. We determined the depth of all affine monoids of degrees four through six and obtained the following

depth\d	4	5	6
1	169	1899	23507
2	295	2171	9167
3	48	26	94

The numbers for depth 3 (Cohen-Macaulay) agree with Corollary 4.9. For depth 2 a finer breakdown, giving the number of curves with dir = 1,2, or 3 directions of arms (over all congruence classes) is given by the table

dir\d	4	5	6
1	144	261	1464
2	135	960	4791
3	16	950	2912

There is not enough data for a clear trend, and indeed we expect the number of divisors of d to play an important role. However the cases of Theorems 3.6 (arms in 3 directions) and 3.7 (arms in one or two directions) both occur in some abundance. We have also noted that there are 216 (out of 512) cases where $S' = \mathbb{N}^3_d$ for $d = 4$, 3375 (out of 4096) for $d = 5$, and 19683 (out of 32768) for $d = 6$, again without a clear trend.

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