LANGLANDS RECIPROCITY FOR CERTAIN GALOIS EXTENSIONS

PENG-JIE WONG

ABSTRACT. In this note, we study Artin’s conjecture via group theory, and derive Langlands reciprocity for certain solvable Galois extensions of number fields, which extends the previous work of Arthur and Clozel. In particular, we show that all nearly nilpotent groups and all groups of order less than 60 are of automorphic type.

1. INTRODUCTION

Nearly a century ago, Emil Artin [2] introduced a new kind of L-function, which generalises both Dirichlet L-functions and Hecke L-functions, and conjectured that all his L-functions attached to non-trivial irreducible characters can be extended to entire functions. Via his celebrated reciprocity law, Artin showed that all his L-functions attached to character of degree 1 correspond to Hecke L-functions and then established his conjecture in this case.

After Artin, Langlands [18] and Tunnell [26] proved Artin’s conjecture for any 2-dimensional irreducible representation with solvable image, and it is a major result in the Langlands program. More recently, the case of odd 2-dimensional irreducible representations (of the absolute Galois group of $\mathbb{Q}$) with non-solvable images was settled by Khare and Wintenberger [15]. Moreover, in light of Artin’s work, Langlands conjectured that all Artin L-functions are automorphic, which is sometimes called the strong Artin conjecture or the Langlands reciprocity conjecture.

In a different vein, Brauer [5] showed that every Artin L-function admits a meromorphic continuation via his induction theorem. Also, it is well-known that Artin’s conjecture is true for any supersolvable Galois extension of number fields. This follows from the fact that supersolvable groups are M-groups, the groups whose irreducible characters are monomial. Undoubtedly, these results suggest that the group-theoretic method shall play a role in studying the (strong) Artin conjecture. In fact, by knowing that all subgroups of nilpotent groups are subnormal, the Arthur-Clozel theory, which will be discussed in Section 2, implies that Langlands reciprocity holds for all nilpotent Galois extensions of number fields.

For non-nilpotent cases, Langlands reciprocity has been derived for certain solvable Frobenius extensions by Zhang [28], which will be discussed in the next section. More recently, Langlands reciprocity is established for $A_4$, $S_4$, $SL_2(\mathbb{F}_3)$, and $GL_2(\mathbb{F}_3)$-extensions. The first two cases were proved by Cho in his PhD thesis [6] (although Cho [6] said these two cases are well-known for experts). Indeed, Cho derived his theorem based on the work of Kim [16] on $SL_2(\mathbb{F}_3)$ and $GL_2(\mathbb{F}_3)$-extensions.

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In this note, we will apply a method of “low dimensional groups” developed by
the author in [27] to study the strong Artin conjecture. (We call a group low di-
mensional if ALL its irreducible characters are of “small degree”.) We will say a
finite group $G$ is nearly nilpotent if it admits a normal subgroup $N$, all of whose
irreducible characters are of degree less than or equal to 2, such that $G/N$ is nilpo-
tent. These groups will be shown to be solvable and discussed in Sections 2 and 6.
In addition, as a theorem of Shafarevich asserts every finite solvable group is realis-
able over $\mathbb{Q}$, our results below really present an enlargement of Galois extensions of
number fields satisfying Langlands reciprocity.

Theorem 1.1. Let $K/k$ be a Galois extension of number fields with Galois group $G$. If $G$ is
either a direct product of two nearly nilpotent groups or a group of order less than 60,
then Langlands reciprocity is true for $K/k$.

Theorem 1.2. Suppose that $K/k$ is a solvable Frobenius Galois extension with Galois group
$G$ (with a Frobenius complement $H$). Then the Artin conjecture holds for $K/k$. Let $F(H)$ be
the maximal normal nilpotent subgroup of $H$. If either $H/F(H)$ is not isomorphic to $S_3$ or a
Sylow 2-subgroup of $F(H)$ is abelian, then the Langlands reciprocity law is valid for $K/k$.

2. Reciprocity Law for Certain Solvable Extensions

Throughout this note, $G$ always denotes a finite group, and $H$ and $N$ denote a
subgroup and a normal subgroup of $G$, respectively. We let $Z(G)$ denote the centre
copies of $G$ will be denoted as $G^k$ or $(G)^k$. The maximal normal nilpotent subgroup
of $G$, the Fitting subgroup of $G$, is denoted by $F(G)$. The cyclic group of order $m$
will be denoted as $C_m$. We also let $Irr(G)$ be the set of irreducible characters of $G$, and
$cd(G) := \{\chi(1) | \chi \in Irr(G)\}$. The trivial group will often be denoted by 1. We will
usually let $p$ and $q$ denote primes without mentioning.

For any $\chi \in Irr(G)$, $\chi$ is said to be of automorphic type if for every Galois exten-
sion $K/k$ of number fields with Galois group $G$, Langlands’ reciprocity conjecture
holds for the Artin L-function $L(s, \chi, K/k)$. In addition, $G$ is of automorphic type if
every irreducible character of $G$ is of automorphic type.

We recall that a finite group $G$ is said to be a Frobenius group if there is a non-
trivial proper subgroup $H$ of $G$ such that $g^{-1}Hg \cap H = 1$ whenever $g \in G \setminus H$. In this
case, $H$ is called a Frobenius complement of $G$. In [28], Zhang showed that certain
Frobenius extensions satisfy the Langlands reciprocity conjecture as follows.

Theorem 2.1. [28] Let $K/k$ be a Galois extension of number fields with Galois group $G$.
Assume that $G$ is a Frobenius group and $H$ is a Frobenius complement of $G$. Let $F(H)$ be
the maximal normal nilpotent subgroup of $H$. If $H/F(H)$ is nilpotent, then every Artin
L-function attached to an irreducible representation of $G$ are automorphic over $k$.

Zhang’s method employs the theory of Frobenius groups and nilpotent groups as
well as Arthur-Clozel’s theory of base change and automorphic induction. In light
of Zhang’s work, we will derive several results which enlarge the class of groups
being of automorphic type. Before we state and prove our results, we shall review
some concepts of SM-groups as well as the main theorem of Arthur-Clozel’s theory.
Theorem 2.2. [1, Arthur and Clozel] Let $K/k$ be a cyclic Galois extension of number fields of prime degree, and $\pi$ and $\Pi$ denote automorphic representations induced from cuspidal of $GL_n(\mathbb{A}_k)$ and $GL_n(\mathbb{A}_K)$ respectively (or, in particular, cuspidal automorphic representations of $GL_n(\mathbb{A}_k)$ and $GL_n(\mathbb{A}_K)$ respectively). Then the base change $B(\pi)$ of $\pi$ and the automorphic induction $I(\Pi)$ of $\Pi$ exist. Moreover, $I(\Pi)$ is induced from cuspidal.

Definition 1. [9, Definition 2.3] Let $G$ be a finite group and $N$ be a normal subgroup of $G$. A character $\chi$ of $G$ is called a relative SM-character with respect to $N$ if there exist a subnormal subgroup $H$ with $N \leq H \leq G$ and an irreducible character $\psi \in \text{Irr}(H)$ such that $\text{Ind}_H^G \psi = \chi$ and $\psi|_N \in \text{Irr}(N)$. If every irreducible character of $G$ is a relative SM-character with respect to $N$, then $G$ is said to be a relative SM-group with respect to $N$.

Since all subgroups of a nilpotent group are monomial and subnormal, it is clear that all nilpotent groups are relative SM-groups with respect to the trivial group. Indeed, this fact together with the above-mentioned Arthur-Clozel theorem asserts that all nilpotent groups are of automorphic type, which is predicted by Langlands’ reciprocity conjecture. We refer the interested reader to [1] for the complete details. Also we will give a proof later by invoking below a result of Horváth that gives a sufficient condition for groups being relative SM-groups.

Theorem 2.3. [9, Proposition 2.7] Let $G$ be a finite group and $N$ be a normal subgroup of $G$ such that $G/N$ is nilpotent. Then $G$ is a relative SM-group with respect to $N$.

We remark that in [9], Horváth considered the relation among subgroup-closed M-groups, the groups whose all subgroups are M-groups, SM-groups, and supersolvable groups, and showed that these classes are all distinct.

In the proof of [28, Proposition 4], Zhang used the theory of relative M-groups, which will be discussed in the next section, to show that certain solvable groups are of automorphic type. However, lacking of “subnormality”, it might not be possible to apply the Arthur-Clozel’s theory. Thanks to the above theorem of Horváth, we are able to overcome this obstacle. Moreover, by invoking below a result of Isaacs, we can drop the assumption on the solvability of groups in Zhang’s theorem.

Theorem 2.4. [10, Theorems 12.5, 12.6 and 12.15] If $G$ is a finite group with $|\text{cd}(G)| \leq 3$, then $G$ must be solvable.

Theorem 2.5. Let $K/k$ be a Galois extension of number fields with Galois group $G$. If there is a normal subgroup $N$ of $G$ such that

1: $G/N$ is nilpotent, and
2: all irreducible characters of $N$ are of dimension 1 or 2,

then Langlands’ reciprocity law is valid for $K/k$.

Proof. First of all, by Isaacs’ theorem, $N$ must be solvable. As $G/N$ is nilpotent, $G/N$ is also solvable. Thus, $G$ is necessarily solvable.

We now prove this theorem by using mathematical induction on the order of $G$. Since $G/N$ is nilpotent, Theorem 2.3 asserts that $G$ is a relative SM-group with respect to $N$. Thus, for every $\chi \in \text{Irr}(G)$, there exist a subnormal subgroup $N \leq H$ of $G$ and an irreducible character $\psi \in \text{Irr}(H)$ such that $\text{Ind}_H^G \psi = \chi$ and $\psi|_N \in \text{Irr}(N)$. 


If $H$ is a proper subgroup of $G$, as $H$ clearly satisfies conditions 1 and 2, one can apply the induction hypothesis on $H$. In particular, $\psi$ is automorphic over the fixed field $K^H$, i.e., there is a cuspidal automorphic form $\Pi$ of $GL_{\psi(1)}(K_{K^H})$ such that

$$L(s, \psi, K/K^H) = L(s, \Pi).$$

On the other hand, since $H$ is a subnormal subgroup of $G$, there is an invariant series

$$H = H_0 \leq H_1 \leq \cdots \leq H_{k-1} \leq H_k = G,$$

where for each $i$, $H_i$ is a normal subgroup of $H_{i+1}$. As $G$ is finite, we may require each $H_{i+1}/H_i$ is a (finite) simple group. Since $G$ is solvable, none of these quotient groups can be non-cyclic. Thus, each $H_{i+1}/H_i$ is a cyclic group of prime order. Now applying the Arthur-Clozel theorem of automorphic induction successively, one can derive that $\text{Ind}_{H}^{G} \psi$ corresponds to an automorphic form over $k$. More precisely, there is an automorphic form $\pi$ of $GL_{\chi(1)}(A_k)$ such that

$$L(s, \Pi) = L(s, \pi).$$

Finally, since $\chi$ is irreducible, a result of Jacquet and Shalika (see [14, Theorem 4.7]) asserts that $\pi$ is necessarily cuspidal, which completes the proof in this case.

Otherwise, suppose $H = G$. Since $\chi|_N = \psi|_N \in \text{Irr}(N)$ and all irreducible characters of $N$ are of dimension 1 or 2, $\chi(1)$ is less than or equal to 2. Now Artin reciprocity together with the theorem of Langlands-Tunnell ensures that $\chi$ is automorphic over $k$. □

As one can tell, this theorem indeed implies that all nilpotent groups are of automorphic type, which is the above-mentioned result of Arthur-Clozel. Now let us borrow the following results of Gow and Jacquet-Piatetski-Shapiro-Shalika that will enable us to obtain a slight improvement of Theorem 2.5.

**Theorem 2.6.** [8] Let $\chi$ be an irreducible character of odd degree of a solvable group $G$. If $\chi$ is real-valued, then $\chi$ is monomial and rational-valued.

**Theorem 2.7.** [13] Let $K/k$ be a non-normal cubic extension of number fields. Let $\chi$ be an idèle class character of $K$. Then the automorphic induction $I(\chi)$ of $\chi$ exists as an automorphic representation of $GL_3(A_k)$.

Combining the Arthur-Clozel theory and these two theorems together gives:

**Theorem 2.8.** Let $G$ be a solvable group, and let $\rho$ be an irreducible representation of $G$ of dimension 3 whose character is real-valued. Then $\rho$ is of automorphic type.

Using a similar argument as in the proof of Theorem 2.5, the above-mentioned theorem of Isaacs, and this result, we then derive:

**Theorem 2.9.** Assume that $G$ has a normal subgroup $N$ satisfying

1: $G/N$ is nilpotent, and
2: all irreducible characters of $N$ are of dimension $\leq 3$.

Suppose, further, that all 3-dimensional irreducible characters (if any) of subnormal subgroups of $G$ containing $N$ are real-valued. Then $G$ is of automorphic type.
Following Ramakrishnan [21], we let $GO(n, \mathbb{C})$ denote the subgroup of $GL(n, \mathbb{C})$ consisting of orthogonal similitudes, i.e., matrices $M$ such that $M^t M = \lambda_M I$, with $\lambda_M \in \mathbb{C}$, and we will say that a $\mathbb{C}$-representation $(\rho, V)$ (of the absolute Galois group of a number field $F$) is of $GO(n)$-type if and only if $\dim V = n$ and it factors as $\rho : Gal(\overline{F}/F) \to GO(n, \mathbb{C}) \subset GL(V)$.

In his paper [21], Ramakrishnan derived the modularity of solvable Artin representations of $GO(4)$-type as follows.

**Theorem 2.10.** Let $F$ be a number field and let $\rho$ be a continuous, 4-dimensional representation of $Gal(\overline{F}/F)$ whose image is solvable and lies in $GO(4, \mathbb{C})$. Then $\rho$ is modular.

Applying this theorem together with an analogous argument as before then gives:

**Theorem 2.11.** Let $G$ be a group. Suppose that there is a normal subgroup $N$ of $G$ so that $1$: $G/N$ is nilpotent, and

$2$: all irreducible characters of $N$ are of dimension either 1, 2, or 4.

If all 4-dimensional irreducible representations (if any) of subnormal subgroups of $G$ containing $N$ are of $GO(4)$-type. Then $G$ is of automorphic type.

By the theory of Frobenius groups, if $G$ is a Frobenius group with a Frobenius complement $H$, there exists a normal subgroup $N$ of $G$ such that $G = N \rtimes H$, where $N$ is called a Frobenius kernel of $G$. Moreover, one has below a theorem.

**Theorem 2.12.** Let $N$ be a a Frobenius kernel of a Frobenius group $G$. For $\chi \in Irr(G)$ with $N \nsubseteq Ker\chi$, one has $\chi = Ind_{N}^{G} \psi$ for some $\psi \in Irr(N)$.

On the other hand, all Sylow subgroups of a Frobenius complement are cyclic or generalised quaternion groups. Furthermore, a deep theorem of Thompson asserts that every Frobenius kernel must be nilpotent. For more details, we refer the interested reader to [10, Chapter 7]. From this, we have the following theorem.

**Theorem 2.13.** Let $K/k$ be a Galois extension of number fields with Galois group $G$. Suppose $G = N \rtimes H$ is a Frobenius group with a Frobenius kernel $N$ and a Frobenius complement $H$. If $H$ is solvable and of automorphic type, then so is $G$.

**Proof.** Let $\chi$ be an irreducible character of $G$. If $Ker\chi$ contains $N$, then $\chi$ can be seen as an irreducible character of $H$. As $H$ is of automorphic type, $\chi$ is automorphic over $k$.

If $N \nsubseteq Ker\chi$, then by Theorem 2.12, there is $\psi \in Irr(N)$ such that $\chi = Ind_{N}^{G} \psi$. Since $N$ is nilpotent, $N$ is of automorphic type. In addition, $K^{N}/k$ is a solvable Galois extension, Arthur-Clozel’s theory yields that $\chi$ is automorphic over $k$. \qed

Now suppose that $G$ is a Frobenius group and $H$ is a Frobenius complement of $G$. Assume, further, that the Fitting subgroup $F(H)$ of $H$ satisfies that $H/F(H)$ is nilpotent. As every Sylow subgroup of $H$ is either cyclic or a generalised quaternion group, all irreducible characters of $F(H)$ are of degree 1 or 2. Thus, Theorems 2.5 and 2.13 assert that $G$ is of automorphic type, which is exactly Theorem 2.1.

We now recall a result that gives a non-trivial bound of character degrees.
Lemma 2.14. [10, pp. 28] Let $G$ be a finite group and $Z(G)$ its centre. Then for every irreducible character $\chi$ of $G$, one has

$$\chi(1)^2 \leq [G : Z(G)].$$

We now give some sufficient conditions for (solvable) groups being of automorphic type. First of all, as any nilpotent group is isomorphic to a direct product of its Sylow subgroups and the derived subgroup of any supersolvable group is nilpotent, we then have the following corollary.

Corollary 2.15. If $G$ is a supersolvable group of order $2^n p_1^{n_1} \cdots p_k^{n_k}$ with $n_i \leq 2$ and $n \leq 4$, then $G$ is of automorphic type.

In addition, since all Z-groups, the groups whose all Sylow subgroups are cyclic, are supersolvable, a moment’s reflection shows:

Corollary 2.16. All Z-groups are of automorphic type. In particular, all groups of square-free order are automorphic type.

We recall that a group $G$ is said to be abelian-by-nilpotent if $G$ admits an abelian normal subgroup $A$ with $G/A$ nilpotent. By a result of Huppert, all abelian-by-nilpotent groups are M-groups. Thanks to Theorem 2.5, one can easily conclude that every abelian-by-nilpotent group is even of automorphic type.

This section will close with some semi-numerical theorems, which in particular, present a simple proof for Cho and Kim’s automorphy results of $A_4$, $S_4$, and $SL_2(F_3)$-extensions. For $GL_2(F_3)$, which have been shown to be of automorphic type by Kim, we will treat it in the next section via our method of low dimensional groups.

Theorem 2.17. Let $p$ and $q$ be distinct primes. If $G$ is of order $pq$, $p^2q$, or $p^2q^2$, then $G$ is of automorphic type.

Proof. By the Sylow theorems, $G$ must have a normal Sylow subgroup $N$ (see, for example, [11, Theorems 1.30 and 1.31] and [24, 6.5.2]). It is clear that $N$ must be abelian, and that $G/N$ is either a $p$-group or a $q$-group. Now the claim follows from Theorem 2.5 immediately. \hfill $\square$

Theorem 2.18. Let $p$ be an odd prime. If $G$ is of order $8p$, then $G$ is of automorphic type.

Proof. Again the Sylow theorems asserts $G$ admits a normal Sylow subgroup $N$ unless $G \cong S_4$ (see [11, Theorems 1.32 and 1.33]). Assuming that $G$ is not isomorphic to $S_4$, since all irreducible characters of $N$ are of degree $\leq 2$, and $G/N$ is clearly nilpotent, Theorem 2.5 yields $G$ is of automorphic type.

Now suppose $G$ is isomorphic to $S_4$. Then $cd(G) = \{1, 2, 3\}$. Since all irreducible characters of $S_4$ are rational-valued, the Artin-Langlands-Tunnell theorem and Theorem 2.8 assert that $G$ is of automorphic type. \hfill $\square$

3. NEARLY SUPERSOLVABLE GROUPS AND NEARLY MONOMIAL GROUPS

We recall that Taketa’s theorem [25] asserts that (finite) M-groups are necessarily solvable. In addition, Taketa’s theorem can be generalised (see, for example, [3, Chapter 14]) for groups whose irreducible characters are all induced from $n$-dimensional characters with $n \leq 2$. We will call these groups nearly monomial (or
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NM for short). Thanks to the theorem of Artin-Langlands-Tunnell and the generalisation of Taketa’s theorem, Artin’s conjecture holds for every Galois extension of number fields whose Galois group is an NM-group. In light of Theorem 2.5, we will introduce the notion of nearly supersolvable groups.

**Definition 2.** A finite group $G$ is said to be nearly supersolvable (or NSS for short) if it has a normal subgroup $N \in C$ such that $G/N$ is supersolvable, where $C$ denotes the class consisting of groups whose irreducible representations are of dimension $\leq 2$.

By results of Amitsur and Isaacs (see Theorem 2.4), all groups belonging to $C$ are necessarily solvable. Therefore, nearly supersolvable groups are indeed solvable. On the other hand, it is clear that all supersolvable groups are nearly supersolvable. For more properties of NSS-groups, we refer the interested to [27]. However, for the benefit of the reader, we recall some concepts of relative M-groups and a result concerning NSS-groups.

**Definition 3.** [10, Definition 6.21] Let $G$ be a finite group and $N$ be a normal subgroup of $G$. A character $\chi$ of $G$ is called a relative M-character with respect to $N$ if there exist a subgroup $H$ with $N \leq H \leq G$ and an irreducible character $\psi \in \text{Irr}(H)$ such that $\text{Ind}_H^G \psi = \chi$ and $\psi|_N \in \text{Irr}(N)$. If every irreducible character of $G$ is a relative M-character with respect to $N$, then $G$ is said to be a relative M-group with respect to $N$.

**Theorem 3.1.** [10, Theorem 6.22] Let $G$ be a finite group and $N$ be a normal subgroup of $G$ such that $G/N$ is supersolvable. Then $G$ is a relative M-group with respect to $N$.

Via this theorem (and a moment’s reflection), the author shows the following.

**Theorem 3.2.** [27] All nearly supersolvable groups are NM-groups.

Now let us give an variant of Theorem 2.13, and let us further borrow below a structure theorem of Frobenius complements (see, for example, [22, Lemmata 18.3 and 18.4] or [12, Theorems 6.14 and 6.15]). (We note that Frobenius complements are called Frobenius subgroups in [12].)

**Corollary 3.3.** Let $K/k$ be a Galois extension of number fields with Galois group $G$. Suppose $G = N \rtimes H$ is a Frobenius group with a Frobenius kernel $N$ and a Frobenius complement $H$. If Artin’s conjecture is true for $K^H/k$, then Artin’s conjecture holds for $K/k$.

**Theorem 3.4.** If $H$ is solvable Frobenius complement, then $H$ satisfies one of the following.

**Type 1:** $H = SQ$, where $S$ is a normal cyclic subgroup of $H$ and $Q$ is cyclic.

**Type 2:** $H = SQ$, where $S \trianglelefteq H$ is cyclic and $Q$ is a generalised quaternion group.

**Type 3:** $H$ is isomorphic to $SL_2(\mathbb{F}_3)$.

**Type 4:** $H/\text{F}(H) \cong S_3$, where $\text{F}(H)$ is the maximal normal nilpotent subgroup of $H$.

We also remark that the above results indeed enable the author shows that the Artin conjecture is true for all solvable Frobenius extensions in [27]. Now, by Theorems 2.5 and 2.17, and the fact that every Sylow subgroup of a Frobenius complement is either a cyclic or generalised quaternion group, we have below a theorem.

**Theorem 3.5.** Suppose that $K/k$ is a solvable Frobenius Galois extension with Galois group $G$. Then the Artin conjecture holds for $K/k$. Moreover, if a Frobenius complement of $G$ is of Type 1, 2, or 3, then the Langlands reciprocity law is valid for $K/k$. 
Moreover, applying our method of low dimensional groups, we still can say a little more for Frobenius complements of Type 4.

**Theorem 3.6.** If $G$ is a solvable Frobenius group $G$ with a Frobenius kernel $N$ and a Frobenius complement $H$, then any irreducible character $\chi$ of $G$ is of automorphic type unless $N \subseteq \text{Ker}\chi$, $\chi$ is of degree 6 and induced from a character of degree 2, and $H$ is of Type 4.

**Proof.** As we have shown before, if $N \nsubseteq \text{Ker}\chi$, $\chi$ is of automorphic type. Also Theorem 3.5 asserts that if $H$ is not of Type 4, $G$ is of automorphic type. Thus, we may assume $H/F(H) \cong S_3$ and $N \subseteq \text{Ker}\chi$. In this case, $\chi$ can be seen as a character of $\hat{H}$. Since $H/F(H)$ is isomorphic to $S_3$, Theorem 3.1 implies that $\chi$ must be induced from an irreducible character $\psi$ of degree $\leq 2$ of a subgroup $\hat{H} \leq H$ of index 1, 2, 3, 6.

By the Arthur-Clozel theory and Theorem 2.7, it sufficient to show that $\chi$ is automorphic type if $\psi$ is of degree 1 or 2 and $[H : \hat{H}] = 6$. However, as $S_3$ has a subgroup of index 2, there exists a subgroup $\hat{H}$ with $\hat{H} \leq H \leq H$ and $[H : \hat{H}] = 2$. In other words, $[\hat{H} : H] = 3$, and hence Arthur-Clozel’s theory asserts $\text{Ind}_{\hat{H}}^H \psi$ is of automorphic type. Now inducing $\text{Ind}_{\hat{H}}^H \psi$ from $\hat{H}$ to $H$ and applying Arthur-Clozel’s induction again completes the proof. \qed

**Corollary 3.7.** Assume that $G$ is a solvable Frobenius group $G$ with a Frobenius complement $H$. If any Sylow 2-subgroup of the Fitting subgroup $F(H)$ of $H$ is abelian, then $G$ is of automorphic type. In particular, if 16 does not divide $|G|$, then $G$ is of automorphic type.

**Proof.** By Theorem 3.5, we may assume $H/F(H)$ is isomorphic to $S_3$. Observe that if 16 does not divide $|G|$, then 8 cannot divide $F(H)$. In this case, any Sylow 2-subgroup of $F(H)$ is abelian. Since for every $p > 2$, all Sylow $p$-subgroups of $H$ are cyclic and $F(H)$ is nilpotent, $F(H)$ is abelian if any Sylow 2-subgroup of $F(H)$ is.

Now the theory of relative M-groups tells us that all irreducible characters of $H$ are monomial. Thus, Theorem 3.6 (together with its proof) implies $H$ is of automorphic type as there is no $\chi \in \text{Irr}(G)$ such that $N \subseteq \text{Ker}\chi$ and $\chi$ is of degree 6 and induced from a 2-dimensional character simultaneously. \qed

As shown in the proof of Theorem 3.6, we cannot derive the automophy for irreducible characters of degree 6, induced from a character of degree 2. Nevertheless, if the existence of the automorphic induction is assumed, one will have the following.

**Theorem 3.8** (Conditional). If the non-normal cubic automorphic induction exists for all 2-dimensional cuspidal automorphic representations, then all solvable Frobenius groups are of automorphic type.

To end this section, we will apply Theorem 2.7 to derive some sufficient conditions for NSS groups being of automorphic type.

**Theorem 3.9.** Suppose $G$ is NSS. If $G$ has a normal subgroup $N$, whose irreducible characters are of degree $\leq 3$, such that $G/N$ is nilpotent, then $G$ is of automorphic type.

**Proof.** We induct on the order of $|G|$. By Theorem 2.3, $G$ is a relative SM-group with respect to $N$. Thus, for every irreducible character $\chi$ of $G$, there exist a subnormal subgroup $H$ with $N \leq H \leq G$ and an irreducible character $\psi \in \text{Irr}(H)$ such that
Ind_H^G \psi = \chi \text{ and } \psi|_N \in \text{Irr}(N). \text{ If } H \neq G, \text{ then the induction hypothesis assures that } H \text{ is of automorphic type, and so applying Arthur-Clozel’s theory completes the proof in this case.}

Now assume that } H = G. \text{ Since } G \text{ is NSS, } G \text{ is an NM-group, and } \chi \text{ must be induced from a character of degree 1 or 2. On the other hand, as } \chi|_N = \psi|_N \text{ is an irreducible character of } N, \chi \text{ is of degree } \leq 3. \text{ If } \chi(1) \leq 2, \text{ then Artin reciprocity and the Langlands-Tunnell theorem assert that } \chi \text{ is of automorphic type. Otherwise, for } \chi \text{ of degree 3, } \chi \text{ must be a monomial character. Now applying Arthur-Clozel’s theory and Theorem 2.7 completes the proof.} \square

Corollary 3.10. \text{ If } G \text{ is a finite group of order 54 or 162, then } G \text{ is of automorphic type.}

\text{Proof.} \text{ By [24, 7.2.15], } G \text{ is a supersolvable group. Since any Sylow 3-subgroup } P \text{ of } G \text{ has index 2, } P \text{ is a normal subgroup. As all non-trivial p-groups have non-trivial centre, } [P : Z(P)] \leq 27. \text{ Thus, Lemma 2.14 yields that } cdt(P) \subseteq \{1, 3\}. \text{ Since } G/P \text{ is cyclic, the corollary follows from Theorem 3.9 immediately.} \square

Corollary 3.11. \text{ Assume that } G \text{ admits an abelian normal subgroup } N \text{ with } G/N \cong Q, \text{ where } S \text{ is a normal subgroup of } Q \text{ of order 3. If } Q/S \text{ is nilpotent then } G \text{ is of automorphic type. In particular, if } G \text{ has an abelian normal subgroup } N \text{ with } G/N \cong S_3, \text{ then } G \text{ is of automorphic type.}

\text{Proof.} \text{ First we observe that as } Q/S \text{ is nilpotent, } Q/S \text{ is supersolvable. Thus, } Q \text{ is also supersolvable. Since } N \text{ is abelian and } G/N \cong Q, \text{ we conclude that } G \text{ is NSS.}

\text{Now lifting the invariant series } 1 \leq S \leq Q \text{ gives a subgroup } H \text{ with } N \leq H \leq G \text{ and } [H : N] = 3. \text{ Moreover, } G/H \text{ is nilpotent. Now Theorem 3.1 implies that all irreducible characters of } H \text{ are of degree 1 or 3, and hence applying Theorem 3.9 completes the proof.} \square

4. Little Groups

In this section, we will apply the machinery developed in the previous sections to derive the “automorphy” for groups of order less than 60. Clearly, the trivial group is always of automorphic type. On the other hand, by the theorem of Arthur and Clozel, we know that all p-groups are of automorphic type. Hence, if } |G| \text{ belongs to}

{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59} \cup \{4, 8, 16, 32, 9, 27, 25, 49\},

then } G \text{ is of automorphic type. There are 26 classes of groups.

\text{According to Theorems 2.17 and 2.18, any group of order } pq, \text{ p}^3, \text{ p}^2q^2, \text{ or } 8p \text{ for some primes } p \text{ and } q \text{ is of automorphic type (thanks to Artin reciprocity, the Langlands-Tunnell theorem, and Arthur-Clozel’s theory). Thus, if } G \text{ has order 6, 10, 12, 14, 15, 18, 20, 21, } 22, 24, 26, 28, 33, 34, 35, 36, 38, 39, 40, 44, 45, 46, 50, 51, 52, 55, 56, 57, \text{ or } 58, \text{ then } G \text{ is of automorphic type. Here we have 29 classes of groups.}

\text{Now, there are only 4 remaining cases, namely, the groups of order 30, 42, 48, or 54. If } G \text{ is of order 30, 42 or 54, } G \text{ is of automorphic type by Corollaries 2.16 and 3.10.}

\text{For } G \text{ of order 48, } G \text{ has a normal subgroup } N \text{ of order 8 or 16. According to Lemma 2.14, all irreducible characters of } N \text{ are of degree } \leq 2. \text{ Since } G/N \text{ is either}
of order 3 or 6, $G/N$ must be supersolvable. Thus, $G$ is clearly NSS and NM. In addition, if $|G/N| = 3$, Theorem 2.5 asserts that $G$ is of automorphic type.

Now assume $|N| = 8$. As $G$ is an NM-group, Artin reciprocity, the Langlands-Tunnell theorem, and Theorem 2.7 ensure that every irreducible character of $G$ of degree $\leq 3$ is of automorphic type. On the other hand, we note that all irreducible representations of $G$ are of dimension $\leq 4$, which can easily be checked via GAP [7] for instance. By the fact that $G$ is nearly supersolvable, and $[G : N] = 6$, we conclude that if $\chi$ is an irreducible character of degree 4, it must induced from a 2-dimensional character of a subgroup $H$ of $G$ containing $N$. As $[G : H] = 2$, $H$ is a normal subgroup, and so $\chi$ is of automorphic type.

To end this section, we note that our method can be applied to groups of order $\geq 60$ in some cases. For example, if $G$ is of order 60, as a consequence of the Sylow theorem, $G$ is either isomorphic to $A_5$, $A_4 \times C_5$, or $C_{15} \rtimes T$ where $T = C_4$ or $T = (C_2)^2$.

Since $A_4$ is of automorphic type, and $C_5$ is abelian, Artin reciprocity and the functoriality of $GL(n) \times GL(1)$ assert that $A_4 \times C_5$ is of automorphic type. (We however will still give another proof in the last section by developing general criteria.) On the other hand, for the third case, $G$ is clearly of automorphic type thanks to Theorem 2.5. Therefore, we have the following.

**Theorem 4.1.** If $G$ is a non-simple group of order 60, then $G$ is of automorphic type.

Also a straightforward application of Sylow’s theory yields that every group of order $16p$ has a normal Sylow subgroup unless $p = 3$. As a consequence, Lemma 2.14 and Theorem 2.5 assert every group of order $16p$ is of automorphic type unless $p = 3$. As shown above, for $G$ of order 48, $G$ is of automorphic type, and we hence have:

**Theorem 4.2.** If $G$ is of order $16p$, then $G$ is of automorphic type.

Similarly, the Sylow theorem yields that every group $G$ of order $27p$ has a normal Sylow subgroup (see [11, Theorem 1.32]). Moreover, if $p \neq 2$ or $p \neq 13$, $G$ has a normal Sylow $p$-subgroup. Therefore, if $G$ is a group of order $27p$ with $p \neq 13$, Theorem 2.5 and Corollary 3.10 assert that $G$ is of automorphic type.

For $p = 13$, by Burnside’s $p^aq^b$-theorem (or the previous discussion), $G$ must be solvable. In particular, $G$ admits a non-trivial abelian normal subgroup $A$. Thus, $G/A$ is either a 3-group or of order $3^a13$ for some $a \leq 2$. Again, Sylow’s theorem asserts that all groups of order $3^a13$ with $a \leq 2$ must have normal Sylow 13-subgroups, which implies that these groups are supersolvable. Thus, $G/A$ is necessarily supersolvable, and so $G$ is NNS. Now as mentioned above, $G$ has a normal Sylow subgroup $P$ (say). In particular, all irreducible characters of $P$ are of degree $\leq 3$. Theorem 3.9 then gives:

**Theorem 4.3.** If $G$ is of order $27p$, then $G$ is of automorphic type.

5. **Groups with Few Non-linear Irreducible Characters**

As discussed in the previous sections, if a $p$-group $G$ is “small” or with a “big” abelian normal subgroup, $cd(G)$ will have only two elements, namely, 1 and $p$. Meanwhile, as $G$ is a $p$-group, it is of automorphic type. One may wonder if groups
with $|\text{cd}(G)| = 2$ are of automorphic type in general. It can be answered by knowing below a result due to Amitsur (for $m = 2$) and Isaacs-Passman.

**Theorem 5.1.** [10, Theorem 12.5 and Corollary 12.6] Let $G$ be a finite group. Assume that $|\text{cd}(G)| = \{1, m\}$. Then the derived subgroup $G'$ of $G$ is abelian.

**Corollary 5.2.** If $|\text{cd}(G)| = 2$, then $G$ is of automorphic type.

**Proof.** By the above theorem, $G'$ is abelian. As $G/G'$ is also abelian, Theorem 2.5 asserts that $G$ is of automorphic type immediately. $\square$

As all irreducible characters of an abelian group have degree 1, Artin reciprocity asserts that all abelian groups are of automorphic type. In this spirit, one may read the above result as “if a group $G$ has only one non-linear irreducible character, then $G$ is of automorphic type”, and wonder if a similar assertion holds for groups having only 2 non-linear irreducible characters. Thanks to the following result of Berkovich, we will give an affirmative answer.

**Theorem 5.3.** [3, Chapter 31, Theorem 6] Let $G$ be a finite group with only two non-linear irreducible characters. Then $G$ is one of the following groups:

1. extra-special groups of order $3^{1+2m}$;
2. Frobenius groups $(C_p)^a \rtimes C_4^{(p^n-1)}$;
3. the Frobenius group $(C_3)^2 \rtimes Q_8$;
4. $|G| = 2m$, $|\text{Z}(G)| = 4$, $|G'| = 2$; or
5. $(C_p)^m \rtimes C_{2p^m-2}$,

where $Q_8$ is the quaternion group of order 8, and the fifth case is due to [3, Theorem 24.7 (g)].

**Corollary 5.4.** Let $G$ be a finite group with only two non-linear irreducible characters. Then $G$ is of automorphic type.

**Proof.** For case 1, $G$ is a 3-group, and hence is of automorphic type. For the remaining cases, it is clear that $G$ always has an abelian normal subgroup $N$ (say) so that $G/N$ is nilpotent. Thus, the corollary follows from Theorem 2.5 immediately. $\square$

For the case that groups have only 3 non-linear irreducible characters, which is also classified by Berkovich (see [3, Chapter 31, Theorem 9]), it is possible to examine whether such groups are of automorphic type by applying Theorem 2.5. But for the sake of conceptual clarity, we will not do it here. We however consider the other extreme, namely, finite groups in which all the non-linear irreducible characters have distinct degrees. This was considered by Berkovich, Chillag, and Herzog.

**Theorem 5.5.** [4] Let $G$ be a non-abelian finite group whose all non-linear irreducible characters have distinct degrees. Then one of the following holds:

1. $G$ is an extra-special 2-group;
2. $G$ is a Frobenius group of order $p^n(p^n - 1)$ for some prime $p$ with an abelian Frobenius kernel of order $p^n$ and a cyclic Frobenius complement; or
3. $G$ is a Frobenius group of order 72 whose Frobenius complement is isomorphic to the quaternion group of order 8.
Applying this theorem and Corollary 2.13, we then obtain:

**Corollary 5.6.** If $G$ is a non-abelian finite group whose all non-linear irreducible characters have distinct degrees, then $G$ is of automorphic type.

6. **The Principle of Functoriality and Nearly Nilpotent Groups**

As we mentioned in the very beginning, Langlands conjectured that every (irreducible) Galois representation arises from a (cuspidal) automorphic representation. This is in fact a part of what is called the principle of functoriality. The principle of functoriality is the core of the Langlands program, and has countless remarkable consequences. For instance, it is well-known that Artin’s conjecture follows from Langlands’ reciprocity conjecture. One may regard this principle as the holy grail of number theory. However, since we could not afford to give all the necessary definitions to state the principle of functoriality, we direct the serious reader to Langlands’ inspiring article [19], and just discuss a small piece of this beautiful principle.

Let $K/k$ be a Galois extension of number fields with Galois group $G = G_1 \times G_2$. On one hand, for every irreducible character $\chi$ of $G$, there exist irreducible characters $\chi_1$ and $\chi_2$ of $G_1$ and $G_2$, respectively, such that $\chi = \chi_1 \times \chi_2$. On the other hand, assuming $G_1$, $G_2$, and $G$ are all of automorphic type. Then for any Galois extension $K/k$ of number fields with Galois group $G$, there is a cuspidal automorphic representation $\pi$ of $GL_{\chi(1)}(\mathbb{A}_k)$ such that

$$L(s, \chi, K/k) = L(s, \pi).$$

Since for each $i$, $G_i$ is of automorphic type, $\chi_i$ can be seen as a cuspidal automorphic representation $\pi_i$ of $GL_{\chi_i(1)}(\mathbb{A}_k)$ as well. Furthermore, as $\chi = \chi_1 \times \chi_2$, we then derive

$$L(s, \pi) = L(s, \pi_1 \otimes \pi_2).$$

In this spirit, the Langlands program then predicts:

**Conjecture 6.1** (The functoriality of $GL(n) \times GL(m)$). Let $\pi_1$ and $\pi_2$ be cuspidal automorphic representations of $GL_n(\mathbb{A}_k)$ and $GL_m(\mathbb{A}_k)$, respectively. Then $\pi_1 \otimes \pi_2$ is a cuspidal automorphic representation of $GL_{nm}(\mathbb{A}_k)$.

This implies that the direct product of any two groups of automorphic type is of automorphic type; and it is recently proved in the case of $GL(2) \times GL(2)$ by Ramakrishnan in [20] and $GL(2) \times GL(3)$ by Kim and Shahidi in [17].

Now let us consider a direct product $G = G_1 \times G_2$, where $G_1$ and $G_2$ are NM-groups. As discussed above, for every irreducible character $\chi$ of $G$, there exist irreducible characters $\chi_1$ and $\chi_2$ of $G_1$ and $G_2$, respectively, such that $\chi = \chi_1 \times \chi_2$. Since both $G_1$ and $G_2$ are NM-groups, for each $i$, there exist a subgroup $H_i$ of $G_i$ and an irreducible character $\psi_i \in Irr(H_i)$ of degree $\leq 2$ such that $\chi_i = Ind_{H_i}^{G_i} \psi_i$. Thus,

$$\chi = Ind_{H_1 \times H_2}^{G_1 \times G_2} (\psi_1 \times \psi_2).$$

However, now one can see that $\chi$ might not be induced from an irreducible character of degree $\leq 2$. As a consequence, we cannot apply the Langlands-Tunnell theorem to deduce Artin’s conjecture directly. But as $\psi_i$ is still of automorphic type (thanks to Artin reciprocity and the Langlands-Tunnell theorem), if we invoke the functoriality
of $GL(n) \times GL(1)$ and $GL(2) \times GL(2)$, we then still are able to derive the automorphy of $\psi_1 \times \psi_2$. Thus, we have the following.

**Theorem 6.2.** If $K/k$ is a Galois extension of number fields whose Galois group is a direct product of two NM-groups, then Artin’s conjecture is true for $K/k$.

Moreover, by applying the Rankin-Selberg theory developed by Jacquet-Piatetski-Shapiro-Shalika, the above discussion then yields:

**Theorem 6.3.** If $K/k$ is a Galois extension of number fields whose Galois group is a direct product of three (or four) NM-groups, then Artin’s conjecture is true for $K/k$.

In a slightly different vein, since any (finite) direct product of nilpotent groups is nilpotent, the Arthur-Clozel theory implies that the principle of functoriality is valid in this case. Naturally, one may be desired to find some “non-nilpotent” examples. At the end of this note, we will give such examples of the principle of functoriality by showing that certain direct products of groups of automorphic type are again of automorphic type. In light of the discussion in the previous section, we define nearly nilpotent groups as follows.

**Definition 4.** A finite group $G$ is called nearly nilpotent if it has a normal subgroup $N \in C$ such that $G/N$ is nilpotent, where $C$ denotes the class consisting of groups whose irreducible representations are of dimension less than or equal to 2.

Since all subgroups and homomorphic images of a nilpotent group are nilpotent, a moment’s thought shows that all subgroups and homomorphic images of any nearly nilpotent group are also nearly nilpotent. As all nilpotent groups are supersolvable, all nearly nilpotent groups form a “closed” subclass of the class of NSS groups. On the other hand, one can read Theorem 2.5 as below.

**Theorem 6.4.** If $G$ is a nearly nilpotent group, then all subgroups and homomorphic images of $G$ are of automorphic type.

Unlike nilpotent groups, the direct product of two nearly nilpotent groups might not be nearly nilpotent. In fact, by the previous discussion, one even cannot expect this would be a NM-group. We however still have the following substitute.

**Theorem 6.5.** If $G_1$ is a nearly nilpotent group and $G_2$ is an abelian-by-nilpotent group, then $G_1 \times G_2$ is nearly nilpotent group and so is of automorphic type.

**Proof.** Since $G_1$ is a nearly nilpotent group, there is a normal subgroup $N_1$ of $G_1$ belonging to $C$ such that $G_1/N_1$ is nilpotent. On the other hand, $G_2$ has an abelian normal subgroup $N_2$ such that $G_2/N_2$ is nilpotent. Thus, we have an invariant series

$$1 \trianglelefteq N_1 \times N_2 \trianglelefteq G_1 \times G_2.$$

Since $(G_1 \times G_2)/(N_1 \times N_2) \cong (G_1/N_1) \times (G_2/N_2)$, which is a direct product of nilpotent groups, $(G_1 \times G_2)/(N_1 \times N_2)$ is nilpotent. Moreover, as all irreducible characters of $N_1 \times N_2$ are clearly of degree $\leq 2$, $N_1 \times N_2 \in C$. Thus, $G_1 \times G_2$ is nearly nilpotent.

Moreover, by invoking Ramakrishnan’s functoriality of $GL(2) \times GL(2)$, one can show the direct product of two nearly nilpotent groups is still of automorphic type.
Moreover, for each \( i \), \( \psi \) an irreducible character there are irreducible characters \( \chi \) with respect to \( G \) such that

\[
\chi = \chi_1 \times \chi_2.
\]

Moreover, for each \( i \), there exist a subnormal subgroup \( H_i \) (containing \( N_i \)) of \( G_i \) and an irreducible character \( \psi_i \in \text{Irr}(H_i) \) such that \( \chi_i = \text{Ind}_{H_i}^{G_i} \psi_i \) and \( \psi_i|_{N_i} \in \text{Irr}(N_i) \). Thus,

\[
\chi = \text{Ind}_{H_1 \times H_2}^{G_1 \times G_2} (\psi_1 \times \psi_2).
\]

Now on one hand, as \( \psi_1 \) and \( \psi_2 \) are of degree \( \leq 2 \), Artin reciprocity and Langlands-Tunnell’s theorem assert that for each \( i \), (by regrading \( \psi_i \) as an irreducible character of \( H_1 \times H_2 \)) \( \psi_i \) corresponds to a cuspidal automorphic representation of dimension \( \psi_i(1) \) over \( K^{H_1 \times H_2} \). Thus, the functoriality of \( GL(n) \times GL(1) \) and \( GL(2) \times GL(2) \) imply that \( \psi_1 \times \psi_2 \) corresponds to a cuspidal automorphic representation of dimension \( \psi_1(1)\psi_2(1) \) over \( K^{H_1 \times H_2} \). On the other hand, as \( H_1 \times H_2 \) is subnormal in \( G_1 \times H_2 \), and \( G_1 \times H_2 \) is subnormal in \( G_1 \times G_2 \), we can conclude that \( H_1 \times H_2 \) is subnormal in \( G_1 \times G_2 \). Putting everything together, the above-mentioned theorems of Arthur-Clozel and Jacquet-Shalika yield \( \chi \) is of automorphic type.

**Theorem 6.7.** Assume \( G_1 \) is abelian-by-nilpotent and \( G_2 \) is nearly supersolvable. Suppose, further, that \( G_2 \) has a normal subgroup \( N \), whose irreducible characters are of degree \( \leq 3 \), such that \( G_2/N \) is nilpotent. Then \( G_1 \times G_2 \) is NSS of automorphic type.

**Proof.** Since \( G_1 \) is abelian-by-nilpotent, it admits an abelian normal subgroup \( N_1 \) such that \( G_1/N_1 \) is nilpotent. On the other hand, as there is \( N_2 \in \mathcal{C} \) such that \( G_2/N_2 \) is supersolvable, we then have an invariant series

\[
1 \leq N_1 \times N_2 \leq G_1 \times G_2,
\]

where \( N_1 \times N_2 \) belongs to \( \mathcal{C} \). Since \( (G_1 \times G_2)/(N_1 \times N_2) \cong (G_1/N_1) \times (G_2/N_2) \) is a direct product of supersolvable groups, \( (G_1 \times G_2)/(N_1 \times N_2) \) is supersolvable. Thus, \( G_1 \times G_2 \) is NSS. Finally, observing that \( G_1 \times G_2 \) has a normal subgroup \( N_1 \times N \), whose all irreducible characters are of degree \( \leq 3 \), such that \( (G_1 \times G_2)/(N_1 \times N) \) is nilpotent, Theorem 3.9 asserts that \( G_1 \times G_2 \) is of automorphic type.

As one can tell, we in the above proof showed that \( G_2 \) and \( G_1 \times G_2 \) are of the same type, namely, they are both NNS groups that have normal subgroups whose irreducible characters are of degree \( \leq 3 \). However, these groups are essentially NM-groups so that there is nothing new in the view of character theory. The reader may wonder if the above theorem can be improved in the same spirit of Theorem 6.6. To end this note, we present the following generalisation of Theorems 2.9, 3.9 and 6.7 by invoking the functionality due to Ramakrishnan and Kim-Shahidi.

**Theorem 6.8.** Assume that \( G_1 \) is a nearly nilpotent group and that \( G_2 \) has a normal subgroup \( N_2 \), whose irreducible characters are of dimension \( \leq 3 \), such that \( G_2/N_2 \) is nilpotent. Suppose either one of the following conditions is satisfied:
1: All 3-dimensional irreducible characters of every subnormal subgroup of $G_2$ containing $N_2$ are real-valued.

2: $G_2$ is NSS.

Then $G_1 \times G_2$ is of automorphic type.

Proof. As before, there exists $N_1 \in \mathcal{C}$ such that $G_1/N_1$ is nilpotent, and Theorem 2.3 then asserts that $G_1$ is a relative SM-group with respect to $N_1$. Also, for each irreducible character $\chi$ of $G_1 \times G_2$, there are irreducible characters $\chi_1$ and $\chi_2$ of $G_1$ and $G_2$, respectively, such that $\chi = \chi_1 \times \chi_2$. By our assumption on $G_1$ and $G_2$, Horváth's theorem tells us that for each $i$, there exist a subnormal subgroup $H_i$ (containing $N_i$) of $G_i$ and an irreducible character $\psi_i \in \operatorname{Irr}(H_i)$ such that $\chi_i = \operatorname{Ind}^{G_i}_{H_i} \psi_i$ and $\psi_i|_{N_i} \in \operatorname{Irr}(N_i)$. Thus,

$$\chi = \operatorname{Ind}^{G_1 \times G_2}_{H_1 \times H_2} (\psi_1 \times \psi_2),$$

where $\psi_1(1) \leq 2$ and $\psi_2(1) \leq 3$. Thus, $\psi_1 \times 1$ and $1 \times \psi_2$ are of degree less than or equal to 2 and 3, respectively. Since both conditions 1 and 2 imply that $\psi_2$ is a monomial character if $\psi_2(1) = 3$, Arthur-Clozel's theory and Theorem 2.7 yield that $1 \times \psi_2$ is of automorphic type in this case. This fact together with Artin reciprocity and the Langlands-Tunnell theorem asserts that both $\psi_1 \times 1$ and $1 \times \psi_2$ must be of automorphic type. Now observe that $\psi_1 \times \psi_2 = (\psi_1 \times 1) \otimes (1 \times \psi_2)$. The above discussion and the functoriality of $GL(n) \times GL(1)$, $GL(2) \times GL(2)$ and $GL(2) \times GL(3)$ assert that $\psi_1 \times \psi_2$ is also of automorphic type. Finally, as $H_1 \times H_2$ is subnormal in $G_1 \times G_2$, applying Arthur-Clozel's theorem completes the proof. □

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REFERENCES


DEPARTMENT OF MATHEMATICS, QUEEN’S UNIVERSITY, KINGSTON, ONTARIO K7L 3N6, CANADA.
E-mail address: pjwong@mast.queensu.ca