NEARLY SUPERSOLVABLE GROUPS AND APPLICATIONS TO ARTIN L-FUNCTIONS

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ABSTRACT. In this note, we apply the group-theoretic method to study Artin’s conjecture, and introduce the notations of nearly nilpotent groups and nearly supersolvable groups to answer a question of Arthur and Clozel. As an application, we show that Artin’s conjecture is valid for all nearly supersolvable Galois extensions of number fields as well as all solvable Frobenius extensions.

1. INTRODUCTION

In 1923, Emil Artin published his celebrated paper “Über eine neue Art von L-reihen” in which he introduced a new kind of L-function. These L-functions play a role in attacking the Dedekind conjecture, which asserts that the quotient \( \zeta_K(s)/\zeta_k(s) \) of Dedekind zeta functions is entire whenever \( K/k \) is an extension of number fields. Artin conjectured that all his L-functions attached to non-trivial irreducible characters extend to entire functions, and proved his conjecture for the case of all 1-dimensional characters by establishing his reciprocity law.

Let \( K/k \) be a Galois extension of number fields with Galois group \( G \), and \( \chi \) a character of \( G \). The Artin L-function attached to \( \chi \) is denoted by \( L(s, \chi, K/k) \). We recall the Artin L-functions enjoy below three properties:

1: \( L(s, 1_G, K/k) = \zeta_k(s) \),
2: \( L(s, \chi_1 + \chi_2, K/k) = L(s, \chi_1, K/k)L(s, \chi_2, K/k) \),
3: \( L(s, \text{Ind}^G_H \psi, K/k) = L(s, \psi, K/K^H) \),

where \( 1_G \) denotes the trivial character of \( G \), each \( \chi_i \) is a character of \( G \) and \( \psi \) is a character of a subgroup \( H \) of \( G \). From these properties and the orthogonality of irreducible characters, Artin showed that his conjecture implies Dedekind’s conjecture.

Nowadays, there are several theorems concerning this conjecture. For instance, Langlands [14] and Tunnell [25] proved Artin’s conjecture for any 2-dimensional irreducible representation with solvable image, and it is a major result in the Langlands program. Moreover, the case of odd 2-dimensional irreducible representations (of the absolute Galois group of \( \mathbb{Q} \)) with non-solvable images was recently settled by Khare and Wintenberger in their paper “Serre’s modularity conjecture (I)”.

In a different vein, Brauer [6] showed in 1947 that every Artin L-function extends meromorphically via character theory of finite groups. On the other hand, by the work of Aramata and Brauer, Dedekind’s conjecture is valid for any Galois extension of number fields. Moreover, if \( K \) is contained in a solvable normal closure of \( k \),

\[ \zeta_K(s)/\zeta_k(s) \text{ is entire} \]
Uchida [26] and van der Waall [27] independently proved Dedekind’s conjecture in this case. It is worth noting that the proofs of these theorems utilise Artin reciprocity and the theory of monomial representations.

It is well-known that Artin’s conjecture is true for any nilpotent (or supersolvable) Galois extension of number fields. This follows from the fact that supersolvable groups are M-groups, the groups whose irreducible characters are monomial. All these results suggest that the group-theoretic method shall be a key of “optimising” our understanding of the Artin conjecture. Indeed, the Artin conjecture has been derived for certain solvable Frobenius extensions by Zhang [30] by invoking the theory of Frobenius groups.

In [2], Arthur and Clozel derived the Langlands reciprocity, which will be stated briefly in Section 4, for all Galois representations with nilpotent image. A key ingredient of their proof is the fact that all subgroups of nilpotent groups are subnormal. In light of this, Arthur and Clozel asked if one can classify the subnormally monomial groups, i.e., the groups whose all irreducible characters are induced from 1-dimensional characters of subnormal subgroups. To answer this, we will introduce the notion of nearly nilpotent groups in Section 4.

Inspired by the works of Brauer, et al., we are interested in studying the Artin conjecture via the theory of finite groups. The major part of this note is devoted to “collecting” pure group-theoretic results and developing a method of “low dimensional groups”. (We call a group low dimensional if ALL its irreducible characters are of “small degree”.) Such a group-theoretic machinery will allow one to obtain information of representations of groups via their low dimensional (normal) subgroups. Also we will utilise our method to establish the Artin conjecture for certain solvable Galois extensions. For instance, we will show that the Artin conjecture holds for all solvable Frobenius Galois extensions.

We will say a finite group $G$ is nearly supersolvable if it has an invariant series

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_{k-1} \trianglelefteq N_k = G$$

where each subgroup is normal in $G$, the quotient $N_{i+1}/N_i$ is cyclic for every $i \geq 1$, and $N_1$ belongs to the class $C$ consisting of groups whose irreducible representations are of dimension less than or equal to 2. These groups will be shown to be solvable and discussed in Sections 2. In addition, as a theorem of Shafarevich asserts every finite solvable group is realisable over $\mathbb{Q}$, our result below really presents an enlargement of Galois extensions of number fields satisfying Artin’s conjecture.

**Theorem 1.1.** Suppose that $K/k$ is either a solvable Frobenius Galois extension or a nearly supersolvable Galois extension. Then the Artin conjecture holds for $K/k$.

### 2. Nearly Supersolvable Groups and Nearly Monomial Groups

Throughout this note, $G$ always denotes a finite group, and $H$ and $N$ denote a subgroup and a normal subgroup of $G$, respectively. We let $Z(G)$ denote the centre of $G$, and set $G' = [G,G]$ and $G'' = [G',G']$. The maximal normal nilpotent subgroup of $G$, the Fitting subgroup of $G$, is denoted by $F(G)$. The cyclic group of order $m$ will be denoted as $C_m$. We also let $\text{Irr}(G)$ be the set of irreducible characters of $G$,
and $cd(G) := \{ \chi(1) | \chi \in \text{Irr}(G) \}$. The trivial group will often be denoted by 1. We will usually let $p$ and $q$ denote primes without mentioning.

As a consequence of Artin reciprocity, Artin’s conjecture is true for any Galois extension of number fields whose Galois group is a nilpotent group, a supersolvable group, or an M-group. These classes of groups became an area of interest in their own right. For instance, Taketa’s theorem [24] asserts that (finite) M-groups are necessarily solvable. In addition, it is also possible to generalise Taketa’s theorem (see, for example, [4, Chapter 5]) to groups whose irreducible characters are all induced from $n$-dimensional characters with $n \leq 2$. We will call these groups nearly monomial groups (or NM-groups for short).

Thanks to the theorem of Artin-Langlands-Tunnell and the above-mentioned generalisation of Taketa’s theorem, Artin’s conjecture holds for every Galois extension of number fields whose Galois group is an NM-group. Undoubtedly, it is desired to classify the class of NM-groups not only for the purely group-theoretic interest but also for the purpose of studying L-functions.

By a result of Huppert, if a group $G$ admits an abelian normal subgroup $N$ such that $G/N$ is supersolvable, $G$ is an M-group. In light of this, we will introduce the notion of nearly supersolvable groups and discuss some properties of these groups. In fact, one goal of this section is showing that NSS-groups belong to the class of NM-groups. Now we shall start by defining nearly supersolvable groups.

**Definition 1.** A finite group $G$ is said to be nearly supersolvable (or NSS for short) if it has an invariant series of subgroups

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_{k-1} \trianglelefteq N_k = G$$

where each subgroup is normal in $G$, the quotient $N_{i+1}/N_i$ is cyclic for every $i \geq 1$, and $N_1$ belongs to the class $C$ consisting of groups whose irreducible representations are of dimension less than or equal to 2.

We note that the class $C$ is classified by Amitsur as follows.

**Theorem 2.1.** [1, Theorem 15.13] Let $G$ be a finite group. Then all irreducible characters of $G$ are of degree 1 or 2 if and only if either

1: $G$ is abelian,  
2: $G$ has an abelian subgroup of index 2, or  
3: $G/\text{Z}(G) \cong C_2 \times C_2 \times C_2$.

This result has been generalised by Isaacs.

**Theorem 2.2.** [9, Theorems 12.5, 12.6 and 12.15] If $G$ is a finite group with $|cd(G)| \leq 3$, then $G$ must be solvable.

By the above results of Amitsur and Isaacs, all groups belonging to $C$ are necessarily solvable. Therefore, nearly supersolvable groups are indeed solvable. On the other hand, it is clear that all supersolvable groups are nearly supersolvable. In fact, we will see that nearly supersolvable groups behave just like supersolvable groups. To state and prove this formally, we first recall below a lemma that assures that the class $C$ is “closed”.
Lemma 2.3. [21, Chapter 6, Lemma 1.3] Let $G$ be a finite group. Suppose that all irreducible representations of $G$ are of dimension 1 or 2. Assume that $H$ is either a subgroup or homomorphic image of $G$. Then every irreducible representation of $H$ is of dimension 1 or 2.

Theorem 2.4.

1: Every subgroup of an NSS-group is NSS.
2: Every homomorphic image of an NSS-group is NSS. In particular, every quotient group of an NSS-group is NSS.

Proof. Let $G$ be a nearly supersolvable group with an invariant series of subgroups

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_{k-1} \trianglelefteq N_k = G$$

where each subgroup is normal in $G$, the quotient $N_{i+1}/N_i$ is cyclic for every $i \geq 1$, and $N_1$ belongs to the class $C$. Then for any subgroup $H$ of $G$,

$$1 = H \cap N_0 \trianglelefteq H \cap N_1 \trianglelefteq \cdots \trianglelefteq H \cap N_{k-1} \trianglelefteq H \cap N_k = H$$

is an invariant series of $H$ in which each quotient quotient $(H \cap N_i)/(H \cap N_{i-1})$ is isomorphic to the subgroup $(H \cap N_i)/N_{i-1}$ of $N_i/N_{i-1}$.

On the other hand, let $\phi : G \to H$ be a surjective homomorphism, then 

$$1 = \phi(N_0) \trianglelefteq \phi(N_1) \trianglelefteq \cdots \trianglelefteq \phi(N_{k-1}) \trianglelefteq \phi(N_k) = H$$

is an invariant series of $H$. Moreover, for each $i$, $\phi(N_i)/\phi(N_{i-1})$ is a homomorphic image of the quotient group $N_i/N_{i-1}$. Now Lemma 2.3 implies that $H$ is NSS whenever $H$ is a subgroup or homomorphic image of $G$. □

Like supersolvable groups, it is false in general that if both $N$ and $G/N$ are nearly supersolvable, then $G$ is a nearly supersolvable group. However, we have the following weak substitute.

Theorem 2.5. Let $G$ be a group and $N$ its normal subgroup. If $N$ belongs to the class $C$, and $G/N$ is supersolvable, then $G$ is nearly supersolvable.

Proof. By lifting an invariant series of $G/N$ to $G$, we have

$$N = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_{k-1} \trianglelefteq N_k = G$$

where each subgroup is normal in $G$ and every quotient $N_{i+1}/N_i$ is cyclic. Since $N \in C$, extending the above invariant series to the trivial subgroup completes the proof. □

To prove our main theorem, we recall some concepts of relative M-groups.

Definition 2. [9, Definition 6.21] Let $G$ be a finite group and $N$ be a normal subgroup of $G$. A character $\chi$ of $G$ is called a relative M-character with respect to $N$ if there exist a subgroup $H$ with $N \leq H \leq G$ and an irreducible character $\psi \in \text{Irr}(H)$ such that $\text{Ind}_H^G \psi = \chi$ and $\psi | _N \in \text{Irr}(N)$. If every irreducible character of $G$ is a relative M-character with respect to $N$, then $G$ is said to be a relative M-group with respect to $N$.

It can be shown that if $G$ is a finite group and $G/N$ is supersolvable, then $G$ is a relative M-group with respect to $N$. In general, one has the following.
Theorem 2.6. [9, Theorem 6.22] Let $G$ be a finite group and $N$ be a normal subgroup of $G$ such that $G/N$ is solvable. Suppose, furthermore, that every chief factor of every subgroup of $G/N$ is of square-free order. Then $G$ is a relative M-group with respect to $N$.

Now we can state and prove our main theorem.

Theorem 2.7. All nearly supersolvable groups are NM-groups.

Proof. According to the definition, for any nearly supersolvable group $G$, there is an invariant series of subgroups 

$$1 = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_{k-1} \triangleleft N_k = G$$

where each subgroup is normal in $G$, the quotient $N_{i+1}/N_i$ is cyclic for every $i \geq 1$, and $N_1$ belongs to the class $C$ consisting of groups whose irreducible representations are of dimension less than or equal to 2. Quotienting the above invariant series by $N_1$ then gives

$$\langle e \rangle = N_1 \triangleleft N_2 \triangleleft \cdots \triangleleft N_{k-1} \triangleleft N_k = G/N_1,$$

where for each $i \geq 1$, $N_i := N_i/N_1$. According to the third isomorphism theorem, each $N_{i+1}/N_i$ is isomorphic to $N_{i+1}/N_i$, which is cyclic. In other words, $G/N_1$ is supersolvable. Now applying Theorem 2.6, $G$ is a relative M-group with respect to $N_1$. As all irreducible characters of $N_1$ are of degree $\leq 2$, we conclude that $G$ is an NM-group. $\square$

We recall below a result that gives sufficient conditions for groups being supersolvable, which will enable one to obtain some examples of NSS-groups.

Theorem 2.8. [28, pp. 6]

1: If $|G| = qp^n$ and $q|(p-1)$, then $G$ is supersolvable. In particular, if $|G| = 2p^n$, then $G$ is supersolvable.

2: Suppose that $q^2|(p-1)$ and $|G| = qp^n$. Then $G$ is supersolvable. In particular, if $|G| = 4p^n$ with $4|(p-1)$, then $G$ is supersolvable.

By Theorem 2.1, a moment’s reflection shows that all irreducible characters of any group of order $2p$ or $2p^2$ are of degree $\leq 2$. Thus, one has

Theorem 2.9. Let $p$ be a prime. If $|G| = 4p^n$, and $G$ admits a normal subgroup of order 2, 4, 2p or $2p^2$. Then $G$ is NSS. If $|G| = 8p^n$, and $G$ has a normal subgroup of order 4 or 8, then $G$ is NSS. Moreover, if $|G| = 8p^n$ with $4|(p-1)$, and $G$ admits a normal subgroup of order 2, 2p, or $2p^2$, then $G$ is NSS.

To end this section, we give below a sufficient condition for groups of derived length $\leq 3$ being of NSS-groups.

Theorem 2.10. Suppose that $G$ has derived length $\leq 3$. If $G'/G''$ is cyclic, then $G$ is an NNS-group.

Proof. Quotienting the derived series $G$ by $G''$ gives

$$1 \triangleleft G'/G'' \triangleleft G/G''.$$ 

Since $G'/G'$ is abelian, the isomorphism theorem yields that the quotient $(G/G'')/(G'/G'')$
is also abelian. As $G'/G''$ is cyclic, one can conclude that $G/G''$ is supersolvable. Moreover, since $G''$ is abelian, Theorem 2.5 asserts that $G$ is NNS.

3. Nearly Nilpotent Groups

Let $K/k$ be a Galois extension of number fields with Galois group $G$, and $\chi$ an irreducible character of $G$. Inspired by Artin reciprocity, Langlands conjectured that there exists a cuspidal automorphic form $\pi \in \mathfrak{A}(GL_{\chi(1)}(\hat{k}))$ such that

$$L(s, \chi, K/k) = L(s, \pi),$$

where $L(s, \pi)$ denotes the automorphic L-function attached to $\pi$. Now let us assume $\chi = \text{Ind}_H^G \psi$ for some $\psi \in \text{Irr}(H)$, where $H$ is a subgroup of $G$. By the induction invariance property of Artin L-functions, one has

$$L(s, \chi, K/k) = L(s, \text{Ind}_H^G \psi, K/k) = L(s, \psi, K/KH).$$

According to the above Langlands conjecture, there shall be a cuspidal automorphic form $\Pi \in \mathfrak{A}(GL_{\psi(1)}(\hat{k}_H))$ such that

$$L(s, \psi, K/KH) = L(s, \Pi),$$

which suggests that there shall be a map sending $\Pi$ to $\pi$. This conjectural map is called the automorphic induction. As one can tell, the existence of the automorphic induction implies the Langlands reciprocity for all Artin L-functions attached to monomial characters. Indeed, by the theory of Arthur and Clozel [2], such a map exists whenever $G$ is solvable and $H$ is a subnormal group of $G$. From this and the fact that all subgroups of a nilpotent group is subnormal, Arthur and Clozel showed that all Galois representations with nilpotent image are automorphic. As mentioned in the very beginning, it leads them to ask for a classification of subnormally monomial groups, the groups whose all irreducible characters are induced from 1-dimensional characters of subnormal subgroups. We refer the interested reader to [2] for the complete details.

We however note that thanks to the Langlands-Tunnell theorem, the theory of Arthur and Clozel indeed implies that all Artin L-functions attached to characters induced from 2-dimensional characters of subnormal groups are automorphic under a certain solvability condition. In light of this, we are interested in the classification of subnormally NN-groups, which can be seen as a generalisation of Arthur-Clozel’s question, and leads us to consider nearly nilpotent groups as follows.

**Definition 3.** A finite group $G$ is called nearly nilpotent if it has a normal subgroup $N \in C$ such that $G/N$ is nilpotent, where $C$ denotes the class consisting of groups whose irreducible representations are of dimension less than or equal to 2.

Since all subgroups and homomorphic images of a nilpotent group are nilpotent, a moment’s thought shows that all subgroups and homomorphic images of any nearly nilpotent group are also nearly nilpotent. As all nilpotent groups are supersolvable, all NN-groups form a “closed” subclass of the class of NSS-groups. In particular, all NN-groups are solvable. Now let us borrow a results of Horváth, which allows us to (partially) answer the above question of Arthur and Clozel.
**Definition 4.** [8, Definition 2.3] Let $G$ be a finite group and $N$ be a normal subgroup of $G$. A character $\chi$ of $G$ is called a relative SM-character with respect to $N$ if there exist a subnormal subgroup $H$ with $N \leq H \leq G$ and an irreducible character $\psi \in \text{Irr}(H)$ such that $\text{Ind}^G_H \psi = \chi$ and $\psi|_N \in \text{Irr}(N)$. If every irreducible character of $G$ is a relative SM-character with respect to $N$, then $G$ is said to be a relative SM-group with respect to $N$.

**Theorem 3.1.** [8, Proposition 2.7] Let $G$ be a finite group and $N$ be a normal subgroup of $G$ such that $G/N$ is nilpotent. Then $G$ is a relative SM-group with respect to $N$.

Now a moment’s reflection shows below a corollary.

**Corollary 3.2.** If $G$ is nearly nilpotent, then every irreducible characters $G$ is induced from a irreducible character of degree $\leq 2$ of a subnormal subgroup of $G$.

**Remark.** For any $\chi \in \text{Irr}(G)$, $\chi$ is said to be of automorphic type if for every Galois extension $K/k$ of number fields with Galois group $G$, the Langlands reciprocity conjecture holds for the Artin L-function $L(s, \chi, K/k)$. In addition, $G$ is of automorphic type if every irreducible character of $G$ is of automorphic type. In [29], the author shows that any Galois extension of number fields whose Galois group is a direct product of two NN-groups is of automorphic type. It is the main reason why we concentrate on nearly nilpotent groups here.

Now we are in the place to find nearly nilpotent groups. To achieve this, we first recall a result that gives a non-trivial bound of character degrees.

**Theorem 3.3.** [9, pp. 28] Let $G$ be a finite group and $Z(G)$ its centre. Then for every irreducible character $\chi$ of $G$, one has

$$\chi(1)^2 \leq [G : Z(G)].$$

Using this theorem and the fact that every non-trivial $p$-group has the non-trivial centre, we obtain:

**Lemma 3.4.** Let $G$ be a finite group of order 4, 8, or 16. Then all irreducible characters of $G$ are of degree $\leq 2$.

As any nilpotent group is isomorphic to a direct product of its Sylow subgroups, we derive:

**Theorem 3.5.** Let $N$ be a normal nilpotent subgroup of $G$. Suppose that all Sylow $p$-subgroups of $N$ are abelian for $p > 2$, and that all irreducible characters of Sylow $2$-subgroups of $N$ have degree $\leq 2$. If $G/N$ is nilpotent, then $G$ is NN.

Moreover, as the derived subgroup of any supersolvable group is nilpotent, we then have the following corollary.

**Corollary 3.6.** Let $p_1, \ldots, p_k$ be distinct odd primes, and $G$ a group of order $2^n p_1^{n_1} \cdots p_k^{n_k}$ where for each $i$, $n_i \leq 2$ and $n \leq 4$. If $G'$ is nilpotent, then $G$ is NN. In particular, if $G$ is a supersolvable group of order $2^n p_1^{n_1} \cdots p_k^{n_k}$ with $n_i \leq 2$ and $n \leq 4$, then $G$ is NN.

In addition, since all Z-groups, the groups whose all Sylow subgroups are cyclic, are supersolvable, a moment’s reflection shows:

**Corollary 3.7.** All Z-groups are nearly nilpotent.
Corollary 3.8. Let $G$ be a finite group of order $p_1 \cdots p_k$ where $p_1 < \cdots < p_k$ are distinct primes, then $G$ is supersolvable and of automorphic type.

We remark that M. R. Murty-V. K. Murty [17] and M. R. Murty-Srinivasan [18] estimated the number of groups of square-free order, which together with our result provides a significant evident supporting Langlands reciprocity for (solvable) extensions. On the other hand, this corollary implies that all groups of order $pqr$, where $p$, $q$, and $r$ are distinct primes, are NN. As $A_5$ is a simple group of order $60 = 2^2 \cdot 3 \cdot 5$, it is impossible to apply our method to groups of non-square-free order in general. However, by invoking the transfer theory (see, for example, [10, 5C]), we still have the following.

Theorem 3.9. Let $G$ be a finite group of order $p^2qr$, where $2 < p < q < r$ are distinct primes. If $q \nmid (r-1)$, then $G$ is NN.

Proof. Since $|G|$ is not divisible by $p^3$ and $p$ is odd, $G$ has a normal $p$-complement, i.e., $G$ has a normal subgroup $N$ such that $G/N$ has order $p^2$. Now the assumption that $q < r$ and $q \nmid (r-1)$ assures that $N$ is isomorphic to $C_{qr}$. Thus, the corollary follows. \qed

One may invoke the theory of nilpotent numbers to improve the above theorem as follows. Following [22], a positive integer $n = p_1^{a_1} \cdots p_t^{a_t}$, where $p_i$’s are distinct primes, is said to have nilpotent factorization if and only if $p_i^k \not\equiv 1 \mod p_j$ for all integers $i$, $j$ and $k$ with $1 \leq k \leq a_i$. A number $n$ is called a cyclic, abelian, or nilpotent number if all groups of order $n$ are cyclic, abelian, or nilpotent, respectively. Pakianathan and Shankar then derived the following characterisation of these numbers. As mentioned in [22], the problem asking for such a characterisation is not new. Indeed, the cyclic case goes back to Burnside, and the abelian case was asked by Robinson as an “exercise” in his book [19]. Also lower bounds of such numbers have been obtained by M. R. Murty and V. K. Murty in [16].

Theorem 3.10. [22] A positive integer $n$ is a nilpotent number if and only if it has nilpotent factorisation. A positive integer $n$ is an abelian number if and only if $n$ is a cube-free nilpotent number. A positive integer $n$ is a cyclic number if and only if $n$ is a square-free nilpotent number.

Corollary 3.11 (Burnside). A positive integer $n$ is a cyclic number if and only if $n$ and $\phi(n)$ are coprime, where $\phi$ denotes Euler’s totient function.

Putting everything together, we have the following general semi-numerical condition for groups being of nearly nilpotent.

Theorem 3.12. Suppose $G$ is a group of order $p^a m$, where $1 \leq a \leq 2$, $2 < p$ is the smallest divisor of $|G|$, and $p$ and $m$ are coprime. If $m$ is a cube-free nilpotent number, then $G$ is of automorphic type. In particular, if $(m, \phi(m)) = 1$, then $G$ is NN.

Recall that for a field $F$, the general affine group of degree 1 over $F$ is defined as

$$GA_1(F) := F \rtimes F^*,$$

the semidirect product of the additive group of $F$ by the multiplicative group of $F$. Moreover, for $F = \mathbb{F}_q$ with $q = p^k$, $GA_1(\mathbb{F}_q)$ is not nilpotent (unless $q = 2$) but
metabelian. On the other hand, we recall a generalised dihedral group corresponding to an abelian group \( H \) is the (external) semidirect product of \( H \rtimes C_2 \), with the non-identity element acting as the inverse map on \( H \). In particular, all dihedral groups are generalised dihedral groups. Moreover, it is clear that every generalised dihedral group is metabelian. Also, a generalised dihedral group corresponding to \( H \) is nilpotent only if \( H \) is a 2-group.

Thus, the above corollary and discussion then yields below a result giving two infinite families of non-nilpotent groups being of automorphic type.

**Corollary 3.13.** All general affine groups of degree 1 over finite fields and all generalised dihedral groups are NN and so are of automorphic type.

This section will close with some semi-numerical theorems, which imply Cho and Kim’s results on the automorphy of \( A_4 \) and \( SL_2(\mathbb{F}_3) \)-extensions.

**Theorem 3.14.** Let \( p \neq q \) be primes. If \( G \) is of order \( pq, p^2q \), or \( p^2q^2 \), then \( G \) is NN.

**Proof.** By the Sylow theorems, \( G \) must have a normal Sylow subgroup \( N \) (see, for example, [10, Theorems 1.30 and 1.31] and [23, 6.5.2]). It is clear that \( N \) must be abelian, and that \( G/N \) is either a \( p \)-group or a \( q \)-group. Now the claim follows immediately. \( \square \)

**Theorem 3.15.** Let \( p \) be prime. If \( G \) is of order \( 8p \) and not isomorphic to \( S_4 \), then \( G \) is NN.

**Proof.** Again the Sylow theorems asserts \( G \) admits a normal Sylow subgroup \( N \) unless \( G \cong S_4 \) (see [10, Theorems 1.32 and 1.33]). Assuming that \( G \) is not isomorphic to \( S_4 \), since all irreducible characters of \( N \) are of degree \( \leq 2 \), and \( G/N \) is clearly nilpotent, which completes the proof. \( \square \)

**Theorem 3.16.** If \( G \) has order \( p^k q \) or \( p^kq^2 \) with \( p^k < q \), then \( G \) is NN.

**Proof.** The assumption and the Sylow theorems tell us that \( G \) has an abelian normal Sylow \( q \)-subgroup \( Q \) (say). Moreover, \( G/Q \) is clearly a \( p \)-group, and so \( G \) is NN. \( \square \)

We borrow two more results from the theory of transfer homomorphisms (see, for example, [10, Corollaries 5.14 and 5.29]) to derive another variant of Theorem 3.9.

**Theorem 3.17.** Suppose \( G \) is a finite group of order \( p^a m \), where \( p \) and \( m \) are coprime. Assume that no prime divisor \( q \) of \( m \) divides any integer of the form \( p^e - 1 \), where \( 1 \leq e \leq a \). Then \( G \) has a normal \( p \)-complement.

**Theorem 3.18** (Burnside). Suppose \( G \) is a finite group and \( p \) is the smallest prime divisor of \( |G| \). If a Sylow \( p \)-subgroup of \( G \) is cyclic, then \( G \) has a normal \( p \)-complement.

**Theorem 3.19.** Suppose \( G \) is a group of order \( p^a m \), with \( (p, m) = 1 \). Assume either that

1: no prime divisor \( q \) of \( m \) divides any integer of the form \( p^e - 1 \), where \( 1 \leq e \leq a \); or

2: \( p \) is the smallest prime divisor of \( |G| \) and a Sylow \( p \)-subgroup of \( G \) is cyclic.

If \( G \) has an abelian subgroup \( N \) of order \( m \), then \( G \) is NN. In particular, if \( p \) is the smallest prime divisor of \( |G| \), \( a = 1 \), and \( m \) is an abelian number, then \( G \) is NN.
Proof. In each case, the above theorems assert that $G$ admits a normal $p$-complement. In particular, $G$ is $p$-separable. By a theorem of Hall (see, for example, [19, Section 9.1]), all subgroups of $G$ of order $m$ are conjugate. Thus, $N$ must be normal. Since $N$ is abelian and $G/N$ is a $p$-group, the theorem now follows.

Remark. Applying the method developed in this note, the author in [29] showed that Langlands reciprocity holds for all Galois extensions of degree less than 60. This in particular gives a simple proof for $S_4$ and $GL_2(\mathbb{F}_3)$-extensions, which have been shown to be of automorphic type by Kim.

4. AN APPLICATION TO SOLVABLE FROBENIUS EXTENSION

We recall that a finite group $G$ is said to be a Frobenius group if there is a non-trivial proper subgroup $H$ of $G$ such that $g^{-1}Hg \cap H = 1$ whenever $g \in G \setminus H$. In this case, $H$ is called a Frobenius complement of $G$. In [30], Zhang showed that certain Frobenius extensions satisfy the Langlands reciprocity conjecture as follows.

Theorem 4.1. [30] Let $K/k$ be a Galois extension of number fields with Galois group $G$. Assume that $G$ is a Frobenius group and $H$ is a Frobenius complement of $G$. Let $F(H)$ be the maximal normal nilpotent subgroup of $H$. If $H/F(H)$ is nilpotent, then every Artin $L$-function attached to an irreducible representation of $G$ arises from an automorphic representation over $k$. In particular, Artin’s conjecture is true for $K/k$.

Zhang’s method employs the theory of Frobenius groups and nilpotent groups as well as Arthur-Clozel’s theory of base change and automorphic induction. In this section, we will invoke the full-powered theory of solvable Frobenius groups to deduce Artin’s conjecture for all solvable Frobenius extensions. Now we shall start by recalling some concepts of CC-subgroups.

Definition 5. Let $G$ be a finite group. A proper subgroup $N$ is called a CC-subgroup if for every non-trivial element $n$ of $N$, its centraliser $C_G(n)$ is contained in $N$.

Theorem 4.2. [9, Theorem 6.34] Let $N$ be a normal CC-subgroup of a finite group $G$. For $\chi \in \text{Irr}(G)$ with $N \nsubseteq \text{Ker}\chi$, one has $\chi = \text{Ind}_N^G \psi$ for some $\psi \in \text{Irr}(N)$.

Theorem 4.3. Let $K/k$ be a solvable Galois extension of number fields with Galois group $G = NH$, where $N$ is a normal CC-subgroup of $G$. If Artin’s conjecture is true for both $K/K^N$ and $K^H/k$, then Artin’s conjecture holds for $K/k$.

Proof. Let $\chi$ be an irreducible character of $G$.

If $N \nsubseteq \text{Ker}\chi$, then by Theorem 4.2, there is $\psi \in \text{Irr}(N)$ such that $\chi = \text{Ind}_N^G \psi$. Since Artin’s conjecture is true for both $K/K^N$, using the induction invariance property of Artin $L$-functions completes for this case.

If $\text{Ker}\chi$ contains $N$, then $\chi$ can be seen as an irreducible character of $H$. As Artin’s conjecture is valid for $K^H/k$, the theorem follows.

We remark that if $G$ admits a non-trivial proper normal CC-subgroup, then $G$ is a Frobenius group. Conversely, a theorem of Frobenius tells us that if $G$ is a Frobenius group with a Frobenius complement $H$, there exists a normal CC-subgroup $N$ of $G$ such that $G = N \rtimes H$, where $N$ is called a Frobenius kernel $G$. On the other hand, all
Sylow subgroups of a Frobenius complement are cyclic or generalised quaternion groups. Furthermore, a deep theorem of Thompson asserts that every Frobenius kernel must be nilpotent. For more details, we refer the interested reader to [9, Chapter 7]. Now we shall restate Theorem 4.3 as follows.

**Theorem 4.4.** Let $K/k$ be a Galois extension of number fields with Galois group $G$. Suppose $G$ is a Frobenius group with a Frobenius kernel $N$, or, equivalently, $G$ admits a non-trivial proper normal CC-subgroup $N$. If Artin’s conjecture is true for $K^H/k$, then Artin’s conjecture holds for $K/k$.

Let us further borrow below a structure theorem of Frobenius complements (see, for example, [20, Lemmata 18.3 and 18.4] or [11, Theorems 6.14 and 6.15]). (We note that Frobenius complements are called Frobenius subgroups in [11].)

**Theorem 4.5.** Suppose that $H$ is solvable Frobenius complement. Then $H$ satisfies one of the following.

- **Type 1:** $H = SQ$, where $S$ is a normal cyclic subgroup of $H$ and $Q$ is cyclic.
- **Type 2:** $H = SQ$, where $S \triangleleft H$ is cyclic and $Q$ is a generalised quaternion group.
- **Type 3:** $H$ is isomorphic to $SL_2(F_3)$.
- **Type 4:** $H/F(H)$ is isomorphic to $S_3$, where $F(H)$ is the maximal normal nilpotent subgroup, the Fitting subgroup, of $H$.

As every Sylow subgroup of a Frobenius complement is either a cyclic group or a generalised quaternion group, a moment’s reflection together with Theorem 3.14 shows that any Frobenius complement is necessarily NSS. Thus, we have the following theorem.

**Theorem 4.6.** Suppose that $K/k$ is a solvable Frobenius Galois extension. Then the Artin conjecture holds for $K/k$.

To end this note, we remark that our method enables the author in [29] to show that all Frobenius group $G$ with a Frobenius complement of Type 1, 2, or 3 are of automorphic type. Moreover, it can be shown that all irreducible characters $\chi$ of $G$ are of automorphic type unless $\chi$ is of degree 6 and is induced from an irreducible character of degree 2. Nevertheless, if the existence of the automorphic induction is assumed, the author [29] proved the following.

**Theorem 4.7 (Conditional).** If the non-normal cubic automorphic induction exists for all 2-dimensional cuspidal automorphic representations, then all solvable Frobenius groups are of automorphic type.

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