Tony Geramita's notes for his Homological Algebra course, Fall 1994

Books I am consulting

1). Charles A. Weibel - An Introduction to Homological Algebra, Cambridge Studies in Advanced Math., vol. 38

2). P.J. Hilton, U. Stammbach - A Course in Homological Algebra - Spring-Verlag (Graduate Texts in Math.)

3). A. Geramita, C. Small - An Introduction to Homological Methods in Commutative Rings, QPPAM, No. 43

4). D.G. Northcott - An Introduction to Homological Algebra, Cambridge University Press

Homological Algebra

Preliminaries:

Let R be an associative ring with 1 and let U, V and W be three (left) R-modules. Let $f : U \to V$ and $g : V \to W$ be two R-module homomorphisms.

If $g \circ f = 0$ then we say that

$$U \xrightarrow{f} V \xrightarrow{g} W \qquad (*)$$

is a *complex* of R-modules.

Notice that $g \circ f = 0 \Leftrightarrow$ the image of f, $(Im(f))$, $\subseteq Ker(g)$. In this case,

$$Ker(g)/Im(f) = H$$

is an R-module. H is called the *homology* module of the complex (*) above.

Definition: (*) is called an *exact complex* if and only if $H = 0$.

Thus, a homology module "measures " the extent to which a complex is not exact.

We might as well now make the complete definition.

Definition: A sequence $C_\bullet = \{C_n\}_{n\in\mathbb{Z}}$ of (left) R-modules and R-module homomorphisms $d_\bullet = \{d_n\}_{n\in\mathbb{Z}}$, where $d_n : C_n \to C_{n-1}$, is called a *chain complex* of R-modules if $d_{n-1} \circ d_n = 0$ for every n.

The maps d_n are called the *differentials* of the chain complex.

Much of the terminology here comes from homology theory in Algebraic Topology. Thus the elements of the modules C_n are referred to as *n-chains*, the elements of $Ker(d_n)$ as *n-cycles* and the elements of $Im(d_{n+1})$ as *n-boundaries* and

$$H_n = Ker(d_n)/Im(d_{n+1})$$

is called the nth *homology module* of the chain complex.

The complex $C_\bullet$ is called *exact at* C_n if $H_n = 0$ and (simply) an *exact complex* if **all** of its homology modules are 0.

Note: A complex like $(*)$ above can be thought of as a chain complex in many ways by letting $U = C_n, V = C_{n-1}$ and $W = C_{n-2}$ (for any n) and letting $C_i = 0$ otherwise.

Exercise 1.A Let $C_n = \mathbb{Z}/8$ for $n \geq 0$ and let $C_n = 0$ for $n < 0$. Let $d_n : C_n \to C_{n-1}$ be defined by $\bar{x} \to \overline{4x}$ for each n.

Show $\{C_n, d_n\}$ is a chain complex and compute all the homology modules of this complex.

Definition: A sequence of R-modules $C^\bullet = \{C^n\}_{n\in\mathbb{Z}}$ and R-module homomorphisms $\{\partial^n\} = \partial^\bullet$, where

$$\partial^n : C^n \to C^{n+1}$$

is called a *cochain complex* if $\partial^{n+1} \circ \partial^n = 0$ for all n.

This time the elements of $Im(\partial^n)$ are call *coboundaries* and the elements of $Ker(\partial^{n+1})$ are called *cochains.* The quotient module $H^n = Ker(\partial^n)/Im(\partial^{n-1})$ is called the nth *cohomology* module of the cochain complex.

Note: If $\{C_n\}$ is a chain complex and we let $C^n = C_{-n}$ and $\partial^n : C^n \to C^{n+1}$ be defined by $\partial^n = d_{-n}$, then $\{C^n, \partial^n\}$ is a cochain complex.

We say that the complex $C_\bullet$ is *bounded* if almost all the modules C_n are $= 0$. $C_\bullet$ is *bounded above* if there is an integer b such that $C_n = 0$ for all $n \geq b$. $C_\bullet$ is *bounded below* if there is an integer c such that $C_n = 0$ for all $n \leq c$.

If $C_\bullet$ is a bounded complex we usually do not write in all the zero modules.

Consider the complex

$$0 \to A \xrightarrow{f} B \to 0 \quad .$$

This complex is exact at A if and only if f is $1-1$. The complex is exact at B if and only if f is onto. The complex is exact if and only if f is an isomorphism. So, it is possible to speak of properties of functions using the idea of exactness.

Definition: A complex is called a *short exact sequence* if, for some n, it is of the form

$$0 \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \to 0 \qquad (\dagger)$$

and (†) is an exact complex.

It is easy to see that the definition is equivalent to saying: d_n is $1-1$, d_{n-1} is onto and $H_{n-1}=0$.

If $f : M \to N$ is any R-module homomorphism then we have an exact complex:

$$0 \to Ker(f) \xrightarrow{i} M \xrightarrow{f} N \xrightarrow{\eta} N/Im(f) \to 0$$

where i is the inclusion map and η is the canonical map onto a quotient.

Definition: The module $N/Im(f)$ is called the *cokernel* of the map f.

The Snake Lemma:

Given a commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & M' & \overset{\alpha}{\to} & M & \overset{\alpha'}{\to} & M'' & \to & 0 \\ & & f' \downarrow & & f \downarrow & & f'' \downarrow & & \\ 0 & \to & N' & \overset{\beta'}{\to} & N & \overset{\beta}{\to} & N'' & & \end{array} .$$

Then, there is a complex:

$$Ker(f') \xrightarrow{\overline{\alpha}} Ker(f) \xrightarrow{\overline{\alpha'}} Ker(f'') \xrightarrow{\gamma} Coker(f') \xrightarrow{\overline{\beta'}} Coker(f) \xrightarrow{\overline{\beta}} Coker(f'') \qquad (**)$$

which is exact.

Moreover:

$i)$ If α is $1-1$ then $\overline{\alpha}$ is $1-1$,

$ii)$ If β is onto then $\overline{\beta}$ is onto.

Proof: (Mostly left as an exercise - $1A'$) The only nontrivial part of this is the definition of the map γ.

To see how it is defined, let $t'' \in Ker(f'')$, i.e. $f''(t'')=0$. Then, since α' is onto, $t''=\alpha'(t)$ for some $t \in M$. Thus, $f''\alpha'(t)=0=\beta f(t)$. Thus, $f(t) \in Ker(\beta)=Im(\beta')$. I.e. there is a $t' \in N'$ such that $\beta'(t')=f(t)$. We define

$$\gamma(t'') = t' \in Coker(f') .$$

One must show that γ is well-defined and a homomorphism before showing that the sequence $(**)$ is exact.

Corollary: (5-Lemma)

Let

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ 0\downarrow & & f'\downarrow & & f\downarrow & & f''\downarrow & & 0\downarrow \\ 0 & \to & N' & \to & N & \to & N'' & \to & 0 \end{array}$$

be a commutative diagram with exact rows.

Then

1) If f' and f'' are $1-1$ then so is f.
2) If f' and f'' are onto then so is f.
3) If f' and f'' are isomorphisms then so is f.

Proof: I'll leave this proof as an (easy) exercise - $1A''$.

The Group of Homomorphisms

Let M and N be (left) R-modules. We write $Hom_R(M, N)$ to denote the **set** of all R-module homomorphisms from M to N. It is easy to see that this set is an *abelian* group if we let:

$$(f + g)(m) := f(m) + g(m) \ .$$

If, moreover, R is a **commutative** ring, then we can even make this group of homomorphisms into an R-module by:

$$\text{if } \phi \in Hom_R(M, N) \text{ and } r \in R \text{ then } (r\phi)(m) := r(\phi(m)) \ .$$

Exercise 1.B Show that this makes $Hom_R(M, N)$ into an R-module. Also, show that this is not necessarily true if R is not commutative, i.e. give a specific example.

If $\beta : N \to N'$ is an R-module homomorphism, then there is an induced homomorphism of abelian groups, $\beta_* : Hom_R(M,N) \to Hom_R(M,N')$ by

$$\begin{array}{ccc} & & N \\ & \nearrow & \\ M & \beta_* \downarrow & \\ & \searrow & \\ & & N' \end{array}$$

$$\beta_*(\phi) = \beta \circ \phi \ .$$

Exercise 1.C Check that β_* is a homomorphism of abelian groups.

Note further that if:

$$\beta : N \to N' \text{ and } \beta' : N' \to N''$$

then

$$(\beta'\beta)_* : Hom_R(M,N) \longrightarrow Hom_R(M,N'')$$

satisfies

$$(\beta'\beta)_* = (\beta')_* \circ \beta_* \ .$$

Also, if $\beta : N \to N$ is the identity R-module homomorphism then

$$\beta_* : Hom_R(M,N) \to Hom_R(M,N)$$

is the identity group homomorphism.

We think of the discussion above in the following way: Fix an R-module M and consider the symbol $Hom_R(M,-)$. This symbol defines a function from the **set** of (left) R-modules to the set of Abelian groups by

$$Hom_R(M,-) : N \longrightarrow Hom_R(M,N) \ .$$

But, this does more! The "function" also acts on homomorphisms between R-modules i.e.

$$\begin{array}{cccc} & N & & Hom_R(M,N) \\ Hom_R(M,-) : & \beta \downarrow & \longrightarrow & \beta_* \downarrow \\ & N' & & Hom_R(M,N') \end{array}$$

Moreover, this "function" respects composition and identities (the two most important things about functions). Moreover, it maintains the order of composition.

When we make the formal definition later you will that see that we have just shown that $Hom_R(M,-)$ is a covariant functor from the category of (left) R-modules to the category of Abelian groups.

Sect.1B

In a completely analogous way (and I'll leave you to fill in **all** the details) $Hom_R(-,N)$ is also a functor from R-modules to Abelian groups, with one important difference. If we let β^* be the result of applying this functor to $\beta : M \to M'$ then

$$\beta^* : Hom_R(M',N) \longrightarrow Hom_R(M,N)$$

is defined by: $\beta^*(\phi) = \phi \circ \beta$, and

$$(\beta_1 \circ \beta)^* = \beta^* \circ \beta_1^*$$

i.e. the order of composition is reversed. I.e. (as we shall see later) $Hom_R(-,N)$ is a contravariant functor from (left) R-modules to Abelian groups.

We can say more about $Hom_R(M,-)$.

Proposition: Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence of R-modules.

If M is any R-module then

$$0 \to Hom_R(M,A) \xrightarrow{f_*} Hom_R(M,B) \xrightarrow{g_*} Hom_R(M,C) \qquad (\dagger)$$

is also exact.

Proof: (See also exercise 10A) <u>Exactness at $Hom_R(M,B)$:</u>

It is easy to see that if 0 denotes the 0-homomorphism then $0_* = 0$. Since $(g \circ f)_* = g_* \circ f_*$ we get that $(\dagger)$ is a complex. So, it suffices to prove that:

$$Ker(g_*) \subseteq Im(f_*) \ .$$

So, let $\phi \in Ker(g_*)$, i.e. $\phi \in Hom_R(M, B)$ and $g_*(\phi) = 0$, i.e. $g \circ \phi = 0$. Thus, $Im(\phi) \subseteq Ker(g)$.

However, since the original sequence is exact we have $Ker(g) = Im(f)$ and, since f is $1-1$, f gives an isomorphism from A to the $Im(f)$. Thus $\chi = f^{-1} \circ \phi$ is an element of $Hom_R(M, A)$ and $f_*(\chi) = f \circ (f^{-1} \circ \phi) = \phi$, i.e. $\phi \in Im(f_*)$ as we wanted to show.

Exactness at $Hom_R(M, A)$: It suffices to show that f_* is $1-1$. So, suppose that $f_*(\phi) = 0$, $\phi \in Hom_R(M, A)$, i.e. $f \circ \phi = 0$. Since f is $1-1$, ϕ must be the zero map. $\square$

Exercise 2A: Consider the short exact sequence of $\mathbb{Z}$ modules:

$$0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{\eta} \mathbb{Z}/n\mathbb{Z} \to 0 .$$

Apply the functor $Hom_{\mathbb{Z}}(\mathbb{Z}_n, -)$ to this sequence and show that η_* is not onto.

In the light of this proposition we say that the functor $Hom_R(M, -)$ is a *left exact* functor. The exercise shows that, in general, $Hom_R(M, -)$ is not an *exact* functor.

In a completely analogous way we have:

Proposition: Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence of R-modules. If N is any R-module then

$$0 \to Hom_R(C, N) \xrightarrow{g^*} Hom_R(B, N) \xrightarrow{f^*} Hom_R(A, N)$$

is exact.

Exercise 2.B Prove this proposition and then, using the example of Exercise 2.A, show that f^* need not be surjective. This will show that the contravariant functor $Hom_R(-, N)$ is left exact but not necessarily exact.

Direct Sums and Direct Products

Let R be a ring, M, N both R-modules. The *Direct Sum* of M and N is

$$M \oplus N = \{(m, n) \mid m \in M \text{ and } n \in N\}$$

where

$$(m,n)+(m_1,n_1)=(m+m_1,n+n_1) \text{ and } \lambda(m,n)=(\lambda n,\lambda n) \ .$$

There are also two distinguished R-module homomorphisms

$$\iota_M : M \to M\oplus N \ , \ \iota_N : N \to M \oplus N$$

by $\iota_M(m)=(m,0)$ and $\iota_N(n)=(0,n)$.

Note that ι_M and ι_N are $1-1$ and that $\iota_M(M)+\iota_N(N)=M\oplus N$.

Proposition: The triple $\{M\oplus N, \iota_M, \iota_N\}$ satisfies the following universal mapping property:
if V is any R-module and there are maps $\phi_M : M \to V$ and $\phi_N : N \to V$ then there is a unique R-module homomorphism $\phi : M\oplus N \to V$ such that the following diagram commutes:

$$\begin{tikzcd}
 & M \arrow[dl, "\iota_M"'] \arrow[dr, "\phi_M"] & \\
M\oplus N \arrow[rr, dashed, "\phi"] & & V \\
 & N \arrow[ul, "\iota_N"] \arrow[ur, "\phi_N"'] &
\end{tikzcd}$$

Proof: Once stated, the proof is obvious if we let

$$\phi(m,n) := \phi_M(m)+\phi_N(n)$$

Then ϕ satisfies the conditions and the result is immediate. $\square$

Notice that the UMP says that an R-module homomorphism out of $M\oplus N$ is completely determined by its values on $Im(\iota_M)$ and $Im(\iota_N)$.

Notation: In view of the way the map ϕ was constructed we often write $\phi=(\phi_M,\phi_N)$.

There is nothing stopping us from making this definiton in general. Let $\{M_j\}, j\in\mathcal{J}$ be any family of R-modules indexed by the set $\mathcal{J}$.

Definition: The *direct sum* of the modules $\{M_j\}, j\in\mathcal{J}$ is the set

$$\oplus_{j\in\mathcal{J}} M_j = \{(m_j)_{j\in\mathcal{J}} \mid m_j = 0 \text{ for almost all } j\}$$

(where addition and the R-action are defined componentwise) and a family of $1-1$ maps,

$$\iota_{M_j} : M_j \to \oplus_{j\in\mathcal{J}} M_j$$

defined by $\iota_{M_j}(m_j)$ is the infinite tuple which is 0 everywhere except in position j where it is m_j.

This set and the family of maps so defined satisfy a universal mapping property. To wit:

Proposition: Let V be an R-module and suppose there are homomorphisms $\phi_j : M_j \to V$ for each j.

Then there is a unique map $\phi : \oplus_{j\in\mathcal{J}} M_j \to V$ such that $\phi \circ \iota_{M_j} = \phi_j$ for all $j \in \mathcal{J}$.

Proof: We define ϕ as follows

$$\phi((m_j)) = \sum_j \phi_j(m_j) \ .$$

Notice that the sum makes sense because it is finite. The uniqueness follows from the fact that

$$\oplus_{j\in\mathcal{J}} M_j = \sum_{j\in\mathcal{J}} Im(\phi_j)$$

Notation: As with the case above, we denote the uniquely determined map $\phi = (\phi_j)_{j\in\mathcal{J}}$.

There is a similar construction for what is called the *direct product*. To be more precise,

Definition: Let $\{M_j\}_{j\in\mathcal{J}}$ be a family of R-modules. The *direct product* of this family is the R-module

$$\Pi_{j\in\mathcal{J}} M_j = \{(m_j)_{j\in\mathcal{J}} \mid m_j \in M_j\}$$

(with addition and the R action defined componentwise) along with a family of onto mappings

$$\pi_j : \Pi_{j\in\mathcal{J}} M_j \to M_j$$

(called projections) where $\pi_j((m_j)_{j\in\mathcal{J}}) = m_j$.

Note: For $\mathcal{J}$ a finite set the direct sum and the direct product are the same modules, but that is obviously not the case if $\mathcal{J}$ is infinite. One need think only of the difference between polynomials and powerseries!

The direct product also satisfies a Universal Mapping Property.

Proposition: Let V be any R-module and suppose there are R-module homomorphisms $\rho_j : V \to M_j$ for every j.

Then there is a unique map $\rho : V \to \Pi_{j\in\mathcal{J}} M_j$ such that $\pi_j \circ \rho = \rho_j$.

Proof: Let $\rho(v) = (\rho_j(v))_{j\in\mathcal{J}}$. It is an easy matter to verify that this works and is unique. $\square$

Note that this UMP says that a map to the Direct Product is determined if we know what it is when followed by all the projections.

Exercise 2D: Let $M = \oplus_{i=1}^m M_i$ and $N = \oplus_{j=1}^n N_j$ and let $\Phi : M \to N$ be any R-module homomorphism. Show that Φ may be represented by a matrix (ϕ_{ij}) where $\phi_{ij} : M_j \to N_i$.

The notions of direct sum and direct product for modules plays a very fundamental role in our understanding of modules and we shall have occasion to return to these ideas later when we discuss more general categories.

We continue our discussion of some basic ideas about R-modules.

Definition-Proposition: Let M be an R-module and let $S \subseteq M$ be any subset of M.

Then

$$< S >= \{\sum_{s\in S} r_s s \mid r_s \in R \text{ and } s \in S, r_s \neq 0 \text{ for only finitely many } s\}$$

is a submodule of M and is called the *submodule of M generated by S.*

Notes-Definitions:

1). $< S >$ is the smallest submodule of M containing S.

2). If $< S >= M$, then S is said to *generate* M and S is called a *set of generators for* M.

3). If M has a finite set of generators then M is called a *finitely generated* R-module.

4). A set of generators S of M is called a *basis for* M if every element of M can be **uniquely** expressed as a combination of the elements of S.

Exercise 2C: Not every R-module with a finite generating set has a basis. Show that the ideal $I = (x, y) \subset k[x, y]$ does not have a basis.

Definition: The R-module M is called *free* if it has a basis.

Proposition: M is a free R-module with basis S, $\Leftrightarrow$

$$M \simeq \oplus_{s \in S} R_s \text{ where } R_s \simeq R \ .$$

Proof: $\Rightarrow$: Let $m \in M$ then $m = \sum_{s \in S} r_s s$ uniquely with all but a finite number of the $r_s \neq 0$.

Let $\phi : M \to \oplus_{s \in S} R_s$ be given by $\phi(m) = (r_s)_{s \in S}$ where the r_s are the coefficients of m above. By the uniqueness it is easy to see that ϕ is an R-module homomorphism.

Define $\phi_s : R_s \to M$ by $\phi_s(r_s) = r_s s$. By the UMP for direct sums there is a map $\chi : \oplus R_s \to M$ such that $\chi \circ \iota_s = \phi_s$.

Claim: χ and ϕ are inverse to each other.

Proof: $\phi\chi((r_s)) = \phi((\phi_s)((r_s)) = \phi(\sum_{s \in S} r_s s) = (r_s)$.

Also $\chi\phi(\sum r_s s) = \chi((r_s)) = \sum_s r_s s$.

$\Leftarrow$ I'll leave this as a simple exercise.

Sect.2A

Free modules have a nice universal property.

Proposition: Let F be a free R-module on the set S and let M be an R-module.

If f is any <u>set</u> map from S to M, then there is a unique R-module homomorphism $\overline{f} : F \to M$ such that $\overline{f}$ restricted to S is f.

Proof: If we set $f(s) = m_s$ and let

$$\overline{f}(\sum r_s s) = \sum r_s m_s$$

then it is easy to see that $\overline{f}$ is a homomorphism with the desired property and is the only homomorphism extending f to F. $\square$

Proposition: Any R-module M is a quotient of a free module.

Proof: Let F be the free module $F = \oplus_{m \in M} R_m$. Let S be the *standard* basis for F and let $f : S \to M$ be defined by:

$$f(0, \cdots, 0, 1_m, 0, \cdots) = m \ .$$

Clearly f is onto and thus extends to an onto homomorphism $\overline{f} : F \to M$ and so $M \simeq F/Ker(\overline{f})$. $\square$

There is another interesting property that free modules have.

Proposition: Let F be a free module, and consider the following diagram

$$\begin{array}{ccccc} & & F & & \\ & & \downarrow \phi & & \\ M & \xrightarrow{\beta} & N & \to & 0 \end{array}$$

where $\beta : M \to N$ is onto and $\phi : F \to N$ is any homomorphism.

Then, one can find a map $\chi_\phi = \chi : F \to M$ such that $\beta \circ \chi = \phi$.

Proof: Let S be a set which generates F. Then ϕ is determined by its values on S. Since β is onto we can choose $t = \beta^{-1}(\phi(s))$ for every $s \in S$. Define $g(s) = t$. Then $\beta \circ g(s) = \phi(s)$ for every $s \in S$. If we let χ be the homomorphism which extends g to F, then χ works. $\square$

Remark: Coupling this proposition with our earlier remarks we see that if

$$0 \to A \to B \to C \to 0$$

is an exact sequence of R-modules and F is a *free* R-module then

$$0 \to Hom_R(F, A) \to Hom_R(F, B) \to Hom_R(F, C) \to 0$$

is also exact.

Thus, if F is a free R-module we have that $Hom_R(F, -)$ is an **exact** functor.

In the light of this result we make the following definition.

Definition: An R-module P is called *projective* if $Hom_R(P, -)$ is an exact functor.

From what we have already seen, free modules are all projective.

Proposition: $\{P_i\}_{i\in\mathcal{I}}$ are all projective R-modules if and only if $\oplus_{i\in\mathcal{I}} P_i$ is a projective R-module.

Proof: To keep the notation down, I'll only do this for the case of two summands.

$\Rightarrow$: We just need to show that if

$$M \xrightarrow{\beta} N \to 0$$

is exact and

$$\phi : P_1 \oplus P_2 \longrightarrow N$$

then we can "lift " ϕ to M.

Let ι_{P_j}, be the injections given as part of the definition of the direct sum. Then, $\phi \circ \iota_{P_1}$ can be lifted since P_1 is projective. I.e. we have

$$\chi_1 : P_1 \to M \text{ such that } \beta \circ \chi_1 = \phi \circ \iota_{P_1} \ .$$

Similarly we have

$$\chi_2 : P_2 \to M \text{ such that } \beta \circ \chi_2 = \phi \circ \iota_{P_2} \ .$$

Then $\chi = (\chi_1, \chi_2) : P_1 \oplus P_2 \to M$ is the R-module homomorphism guaranteed by the UMP such that $\chi \circ \iota_{P_j} = \chi_j$.

We rewrite

$$\beta \circ \chi_j = \beta \circ (\chi \circ \iota_{P_j}) = \phi \circ \iota_{P_j} \ .$$

I.e. $\beta \circ \chi$ and ϕ are both R-module homomorphisms out of $P_1 \oplus P_2$ which agree after composition with the ι_{P_j} for $j = 1, 2$. By the UMP for direct sums we obtain that $\beta \circ \chi = \phi$ as we wanted to show. $\square$

Definition: The short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is called a *split exact sequence* if there is an R-module homomorphism $h : C \to B$ such that $g \circ h = 1_C$. If it exists, we call h a "splitting".

Example: The exact sequence

$$0 \to A \xrightarrow{\iota_A} A \oplus C \xrightarrow{\pi_C} C \to 0$$

is a split exact sequence with splitting $C \xrightarrow{\iota_C} A \oplus C$.

Proposition: Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a split exact sequence with splitting σ. Then, there is an isomorphism

$$\phi : A \oplus C \longrightarrow B$$

such that we have a commutative diagram

$$\begin{tikzcd} & & B \arrow[dr, "g"] & & \\ 0 \arrow[r] & A \arrow[ur, "f"] \arrow[dr, "\iota_A"'] & & C \arrow[r] & 0 \\ & & A \oplus C \arrow[uu, "\phi"] \arrow[ur, "\pi_C"'] & & \end{tikzcd}$$

Proof: We have $f : A \to B$ and $\sigma : C \to B$ thus, by the UMP for direct sums, there is a unique map $\phi : A \oplus C \to B$ such that

$$i)\ \ f = \phi \circ \iota_A \text{ and } ii)\ \ \sigma = \phi \circ \iota_C \quad .$$

From $i)$ we get the commutativity of the left hand triangle.

To see the commutativity of the right hand triangle, consider

$$g \circ \phi(a, c) = g(f(a) + \sigma(c)) = (gf)(a) + (g \circ \sigma)(c) = 0 + id(c) = c = \pi_C(a, c)$$

as we wanted to show.

Finally, using the 5-Lemma, we see that the map ϕ is an isomorphism. $\square$

We can put all these observations together to get a very nice characterization of projective R-modules.

Theorem: The following are equivalent:

1). P is a projective R-module;

2). The functor $Hom_R(P,-)$ is an exact functor;

3). If $B \xrightarrow{f} P \to 0$ is exact then there is an R-module homomorphism $g : P \to B$ such that $f \circ g = 1_P$;

4). Every short exact sequence ending in P splits;

5). P is a direct summand of every R-module of which it is a quotient;

6). P is a direct summand of a free R-module.

Proof: 1) $\Leftrightarrow$ 2): This is the definition.

2) $\Rightarrow$ 3): We have:

$$0 \to Ker(f) \xrightarrow{i} B \xrightarrow{f} P \to 0$$

exact, so $f_* : Hom_R(P,B) \to Hom_R(P,P)$ is onto. Since $1_P \in Hom_R(P,P)$ there is a $g \in Hom_R(P,B)$ such that $f_*(g) = 1_P$. That is what we wanted.

3) $\Rightarrow$ 4): This is immediate from the definition.

4) $\Rightarrow$ 5): If $M \xrightarrow{f} P \to 0$ is exact then

$$0 \to Ker(f) \to M \xrightarrow{f} P \to 0$$

is exact. From 4) we get that P is a direct summand of M.

5) $\Rightarrow$ 6): Since every module is a homomorphic image of a free module, this is immediate from 5).

6) $\Rightarrow$ 1): P is a direct summand of a free (and hence projective) R-module and hence, by an earlier proposition, P is projective.

$\square$

Exercise 3A:

i). Prove that $\mathbb{Z}_n$ is <u>*not*</u> a projective $\mathbb{Z}$-module.

ii). Using the structure theorem for finitely generated abelian groups show that a f.g. projective $\mathbb{Z}$-module must be free. Hence, for f.g. $\mathbb{Z}$-modules the notions of "projective" and "free" are the same.

iii). Show that if n is not a prime power there always exist f.g. projective $\mathbb{Z}_n$-modules which are not free.

(We'll see later that if n is a prime power then every f.g. projective $\mathbb{Z}_n$-module is free.)

Sect2B

We made the notion of a projective R-module from the functor $Hom_R(M, -)$. With that in mind, we make the following "co-definition".

Definition: An R-module N is called *injective* if $Hom_R(-, N)$ is an exact functor.

From what we have already seen, N is an injective R-module if and only if whenever

$$0 \to A \xrightarrow{f} B$$

is exact, then

$$Hom_R(B, N) \xrightarrow{f^*} Hom_R(A, N) \to 0$$

is also exact.

I.e. given a diagram

$$\begin{array}{ccccc} 0 & \to & A & \xrightarrow{f} & B \\ & & \phi \downarrow & & \\ & & N & & \end{array}$$

there is a map $\chi : B \to N$ making it commute.

To explore this idea we need to extend a theorem from the previous lecture.

Lemma: The short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

splits if and only if there is a map $j : B \to A$ such that $j \circ f = 1_A$.

Proof: $\Rightarrow$: By the previous proposition about split exact sequences we have:

$$\begin{tikzcd} & & B \arrow[dr, shift left, "g"] & & & \\ 0 \arrow[r] & A \arrow[ur, "f"] \arrow[dr, "\iota_A"'] & & C \arrow[ul, shift left, "\sigma"] \arrow[r] & 0 \\ & & A \oplus C \arrow[uu, "\phi"] \arrow[ur, "\pi_C"'] & & \end{tikzcd}$$

where ϕ is an isomorphism, $g \circ \sigma = 1_C$, ι_A and π_C are the maps associated with the direct sum and direct product, and all diagrams commute.

Let $j = \pi_A \circ \phi^{-1}$. Then $j \circ f = \pi_A \circ \phi^{-1} \circ f$. But, $f = \phi \circ \iota_A$, so

$$j \circ f = \pi_A \circ \phi^{-1} \circ (\phi \circ \iota_A) = \pi_A \circ \iota_A = 1_A$$

as we wanted to show.

$\Leftarrow$: Given $j : B \to A$, $g : B \to C$ then, by the UMP for the direct product, we have

$$\Phi : B \to A \oplus C \text{ such that } i)\ \pi_A \circ \phi = j \text{ and } ii)\ \pi_C \circ \Phi = g\ .$$

Consider

$$\begin{tikzcd}
 & & B \arrow[dl, shift left, "j"] \arrow[dr, "g"] \arrow[dd, "\Phi"] & & \\
0 \arrow[r] & A \arrow[ur, "f"] \arrow[dr, "\iota_A"'] & & C \arrow[r] & 0 \\
 & & A \oplus C \arrow[ur, "\pi_C"'] & &
\end{tikzcd}$$

From $ii)$ above, we get the commutativity of the left hand triangle.

Now,

$$(\Phi \circ f)(a) = \Phi(f(a)) := (j(f(a)), g(f(a))) = (a, 0) = \iota_A(a)$$

which, from the previous proposition about split exact sequences, is all we have to show.
$\square$

We know give several characterizations of injective modules, similar (in the "co" sense) to those we gave for projective modules.

Theorem: The following statements are equivalent:

1) N is an injective R-module;

2) If

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is a short exact sequence, then

$$0 \to Hom_R(C, N) \xrightarrow{g^*} Hom_R(B, N) \xrightarrow{f^*} Hom_R(A, N) \to 0$$

is also a short exact sequence.

3) If $0 \to N \xrightarrow{f} B$ is exact then there is an R-module homomorphism $h : B \to N$ such that $h \circ f = 1_N$.

4) N is a direct summand of every module M which contains it.

Proof:

1) $\Leftrightarrow$ 2): This is the definition.

2) $\Rightarrow$ 3): An easy exercise (**Exercise 4A**(5)).

3) $\Leftrightarrow$ 4): trivial

4) $\Rightarrow$ 1): This is the difficult part of the proof and will take quite a while to prove. In order to do this, we will need two lemmas, one easy, one difficult. Let me state the easy one first.

Lemma 1: Let $N = \Pi_{i \in \mathcal{I}} N_i$. Then

$$N \text{ is an injective } R\text{-module} \;\Leftrightarrow\; \text{each } N_i \text{ is an injective } R\text{-module.}$$

Proof: **Exercise 4B** (10)

Lemma 2: Every R-module is contained in an injective R-module.

Proof: Postponed for a bit.

Assuming that the two lemmas are proved, we shall now show how to prove 4) $\Rightarrow$ 1).

Given N, an R-module which is a direct summand of any R-module which contains it, include it in an injective module $\mathfrak{I}$, as guaranteed by Lemma 2. Then $\mathfrak{I} = N \oplus N'$ by assumption. But, by Lemma 1, $\mathfrak{I}$ injective implies that N (and N') is injective and we are done. $\square$

We now return to the problem of finding a proof of Lemma 2. In order to get started on this we first try to get some examples of injective modules. This is non-trivial! We will start by considering injective modules over Principal Ideal Domains (PID's). This includes $\mathbb{Z}$ and $k[x]$, k a field. We start a bit more generally.

Definition: Let R be an integral domain (this means, among other things, commutative with an identity). An R-module, D, is called *divisible* if, for every $d \in D$ and every $0 \neq r$, $r \in R$, there is a $c \in D$ such that $rc = d$ (i.e. we can "divide" by r in D).

Examples:

1) If $R = \mathbb{Z}$ and $D = \mathbb{Q}$, the rationals (under addition), then $\mathbb{Q}$ is a divisible $\mathbb{Z}$-module.
2) If $R = \mathbb{Z}$ and $D = \mathbb{Q}/\mathbb{Z}$ then D is divisible.
3) $\mathbb{Z}$ is **not** a divisible $\mathbb{Z}$-module and neither is $\mathbb{Z}_n$.

Exercise 4C: *i*). (10) No f.g. abelian group is a divisible $\mathbb{Z}$-module.
ii). (10) The direct product of divisible R-modules is again divisible.

We are now ready for our main result about injectives over a PID.

Theorem: Let R be a PID. The R-module M is injective if and only if it is a divisible R-module.

Proof: $\Rightarrow$: (Notice that this direction does not use the fact that R is a PID, only that R is a domain. i.e. This proves: R a domain and M injective $\Rightarrow$ M divisible.)

Suppose that M is injective and let $m \in M$ and let $0 \neq r \in R$. We want to find m' such that $rm' = m$.

Let $\phi : R \to M$ by $\phi(1) = m$. Since R is a free R-module this is enough to well-define ϕ. Similarly, let $\chi : R \to R$ be defined by $\chi(1) = r$. Since R is an integral domain, $ker(\chi) = 0$, i.e. have the following diagram:

$$\begin{array}{ccccc} 0 & \to & R & \xrightarrow{\chi} & R \\ & & \phi \downarrow & & \\ & & M & & \end{array}$$

Since M is injective, there is a map $\delta : R \to M$ such that $\delta \circ \chi = \phi$.

Then, $m = \phi(1) = \delta \circ \chi(1) = \delta(r) = r\delta(1)$. Then $m' = \delta(1) \in M$ is the required element.

$\Leftarrow$: As for the converse, suppose that M is divisible and that we have a diagram

$$\begin{array}{ccccc} 0 & \to & A & \xrightarrow{\mu} & B \\ & & \alpha \downarrow & & \\ & & M & & \end{array}$$

where μ is $1-1$. We need a map $\phi : B \to M$ such that $\phi \circ \mu = \alpha$.

Consider pairs (A_j, α_j) where $A \subseteq A_j \subseteq B$ (we are thinking of "μ" as embedding A in B) and such that $\alpha_j : A_j \to M$ and α_j restricts to α on A. Let Φ be the set of all such pairs.

$$\Phi \text{ is non-empty since } (A, \alpha) \in \Phi \ .$$

Define $(A_j, \alpha_j) \le (A_{j'}, \alpha_{j'})$ if $A_j \subseteq A_{j'}$ and $\alpha_{j'}$ restricted to A_j is α_j.

This ordering is inductive, i.e. every chain has an upper bound (namely $(\cup A_j, \cup \alpha_j)$, where $\cup \alpha_j$ is defined by:

$$\text{if } a \in \cup A_j \ (\text{ then } a \in A_l \text{ for some } l), \text{ and } (\cup \alpha_j)(a) = \alpha_l(a) \ .$$

Using the definition of the ordering, this map is well-defined.

By Zorn's Lemma, Φ contains maximal elements. Let $(\overline{A}, \overline{\alpha}) \in \Phi$ be a maximal element of Φ. If we can show that $\overline{A} = B$ then we are done. So, suppose that $\overline{A} \nsubseteq B$ and let $b \in B$, $b \notin \overline{A}$.

Consider

$$\{r \mid rb \in \overline{A}, r \in R\} = I \ .$$

It is easy to see that I is an ideal of R, and since R is a PID, then $I = (\lambda)$, $\lambda \in R$.

We now try to extend the homomorphism $\overline{\alpha}$ to the module $< \overline{A}, b >$.

Case 1: If $\lambda \neq 0$.

Since $\lambda b \in \overline{A}$ we must have $\overline{\alpha}(\lambda b) = m \in M$. Since M is divisible, $m = \lambda m'$ i.e. $\overline{\alpha}(\lambda b) = \lambda m'$.

We extend $\overline{\alpha}$ to $< \overline{A}, b >$ by:

$$\overline{\alpha}(\overline{a} + sb) = \overline{\alpha}(\overline{a}) + sm' \ .$$

The only question is: Is this extension well-defined, and is it a homomorphism?

Well-defined: I.e. suppose $\overline{a} + sb = \overline{a}_1 + s_1 b$. Is it true that:

$$\overline{\alpha}(\overline{a}) + sm' = \overline{\alpha}(\overline{a}_1 + s_1 m') \ \ ?$$

But, $\overline{a} + sb = \overline{a}_1 + s_1 b$ implies that $\overline{a} - \overline{a}_1 = (s_1 - s)b$. Therefore, $s_1 - s = u\lambda$ for some $u \in R$. I.e. $\overline{a} - \overline{a}_1 = u\lambda b$.

Thus, $\overline{\alpha}(\overline{a}-\overline{a}_1) = \overline{\alpha}(u\lambda b) = u\overline{\alpha}(\lambda b)$. But, $\overline{\alpha}(\overline{a}-\overline{a}_1) = \overline{\alpha}(\overline{a}) - \overline{\alpha}(\overline{a}_1) = u\lambda m' = (s_1 - s)m'$, which is what we wanted to show.

Thus, the map $\overline{\alpha}$ is well-defined. It is easy to see that it is an R-module homomorphism. So, this contradicts the maximality of $(\overline{A}, \overline{\alpha})$, unless $\overline{A} = B$.

Case 2: If $\lambda = 0$.

In this case we do exactly as above, but we just choose $c \in M$ arbitrarily and define: $\overline{\alpha}(\overline{a} + \lambda b) = \overline{\alpha}(\overline{a}) + \lambda c$.

Is this well-defined? Suppose that $\overline{a} + \lambda b = \overline{a}_1 + \lambda' b$, do they have the same image under this extended $\overline{\alpha}$?

But, as before, $\overline{a} - \overline{\alpha}_1 = (\lambda' - \lambda)b$. Since $I = (0)$ we must have $\lambda = \lambda'$. But that, in turn, implies that $\overline{a} = \overline{a}_1$, i.e. the representation is unique and so there is nothing to showing that the extension of $\overline{\alpha}$ is well-defined. It's an easy exercise to show that it is also an R-module homomorphism. $\square$

Exercise 4D:

1). (10) Prove that every quotient of a divisible R-module is again a divisible R-module.

2). (5) If R is a PID then every quotient of an injective R-module is again an injective R-module.

3). (10) The R-module $\mathfrak{I}$ is injective if and only if for every (left) ideal $J \subset R$ and any R-module homomorphism $\phi : J \to \mathfrak{I}$, the following diagram

$$\begin{array}{ccccc} 0 & \to & J & \xrightarrow{\iota} & R \\ & & \phi\downarrow & & \\ & & \mathfrak{I} & & \end{array}$$

can be completed with a map $\chi : R \to I$ such that $\chi \circ \iota = \phi$.

4). (10) Let $R = \mathbb{Z}_m$. Prove that R is an injective R-module.

Sect.3A

Theorem: Every abelian group G can be embedded into an injective abelian group.

Proof: If G is free then $G \simeq \oplus\mathbb{Z}$. Since $\oplus\mathbb{Q}$ is divisible (hence injective) and clearly contains $\oplus\mathbb{Z}$ we are done in this case.

If now G is an arbitrary abelian group then we know that we can find a free abelian group F such that:

$$0 \to K \to F \to G \to 0$$

is exact.

Now $G \simeq F/K \subseteq D/K$, where $F \subseteq D$, D a divisible abelian group. By a previous exercise, D/K is divisible and hence injective as a $\mathbb{Z}$-module. $\square$

Let R be any ring and let C be a right R-module and let G be an abelian group. We can form the abelian group $Hom_{\mathbb{Z}}(C, G)$. But, it is possible to view this abelian group as a left R-module as follows:

$$\text{if } f \in Hom_{\mathbb{Z}}(C, G) \text{ and } r \in R \text{ then } (rf)(c) = f(cr)\ .$$

(I'll leave the verification that i). rf is a $\mathbb{Z}$-module homomorphism and ii). the action of R on $Hom_{\mathbb{Z}}(C, G)$ makes $Hom_{\mathbb{Z}}(C, G)$ into a left R-module , as a (simple) exercise.)

It is also easy to see that if $\phi : C \to D$ is a homomorphism of R-modules then the induced map on abelian groups

$$\phi^* : Hom_{\mathbb{Z}}(D, G) \to Hom_{\mathbb{Z}}(C, G)$$

is an R-module homomorphism. Similarly, if $\chi : G_1 \to G_2$ is a homomorphism of abelian groups then the induced homomorphism of abelian groups

$$\chi_* : Hom_{\mathbb{Z}}(C, G_1) \to Hom_{\mathbb{Z}}(C, G_2)$$

is an R-module homomorphism. (I suggest you fill in the details of this verification.)

The reason for introducing this R-module is apparent from the next Lemma.

Lemma: If D is an injective abelian group then $Hom_{\mathbb{Z}}(R, D)$ is an injective (left) R-module.

Proof: By an earlier exercise, it is enough to show: if L is any left ideal of R and we are given a diagram:

$$\begin{array}{ccccc} 0 & \to & L & \xrightarrow{j} & R \\ & & f \downarrow & & \\ & & Hom_{\mathbb{Z}}(R, D) & & \end{array}$$

then f can be lifted to R.

So, if $x \in L$ then $f(x) : R \to D$ is a $\mathbb{Z}$-module homomorphism. In particular, $(f(x))(r) \in D$.

We can think of f as a function of two variables, $f(x,y)$, where $x \in L$ and $y \in R$. Thought of this way we have:

$$f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2) \ ,$$

$$f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y) \ ,$$

$$f(rx, y) = f(x, yr) \ .$$

Let $g(x) = f(x,1)$ for $x \in L$. This is a map from $L \to D$. It is a $\mathbb{Z}$-module homomorphism. Since D is an injective abelian group, this map can be extended to a homomorphism of abelian groups, g', $g' : R \to D$. We use this map g' to extend f to f', as follows:

$$f'(x,y) = g'(yx) \text{ where } x \in R \text{ and } y \in R \ .$$

Claim: $f'(x,y)$ satisfies the following properties, for x_1, x_2, y_1, y_2, x, $y \in R$:

$i)$ $f'(x_1 + x_2, y) = f'(x_1, y) + f'(x_2, y)$.

$ii)$ $f'(x, y_1 + y_2) = f'(x, y_1) + f'(x, y_2)$,

$iii)$ $f'(rx, y) = f'(x, yr)$ for $r \in R$,

$iv)$ $f'(x,y) = f(x,y) = f(yx, 1) = g(yx)$ for $x \in L$ and $y \in R$.

Proof: Combining $i) - iii)$ gives that we have an R-module homomorphism from R to $Hom_{\mathbb{Z}}(R, D)$ and $iv)$ shows that this homomorphism extends f.

These are all very simple verifications and I leave them to you as an easy exercise to complete the proof of the Lemma. □

Theorem: If M is a left R-module then M can be embedded in an injective (left) R-module.

Proof: First think of M as an abelian group. Then, we have an exact sequence of abelian groups

$$0 \longrightarrow M \xrightarrow{\iota} D$$

where D is an injective abelian group.

It follows that

$$0 \to Hom_{\mathbb{Z}}(R, M) \xrightarrow{\iota_*} Hom_{\mathbb{Z}}(R, D)$$

is also exact.

Claim: $M \simeq Hom_R(R, M) \subseteq Hom_{\mathbb{Z}}(R, M)$, where the isomorphism is of R-modules.

Proof: Clearly, any homomorphism of R-modules is a homomorphism of abelian groups so $Hom_R(R, M) \subseteq Hom_{\mathbb{Z}}(R, M)$.

Define

$$\chi : \ Hom_R(R, M) \longrightarrow M \text{ by } \chi(f) = f(1) \ .$$

Since R is a free R-module, χ is clearly an onto function. Also,

$$\chi(f_1 + f_2) = (f_1 + f_2)(1) = f_1(1) + f_2(1) = \chi(f_1) + \chi(f_2) \ .$$

Moreover,

$$\chi(rf) = (rf)(1) = f(1r) = f(r) = rf(1)$$

this last since f is an R-module homomorphism. Since $rf(1) = r\chi(f)$ we are done with the claim.

The Theorem is an immediate consequence of the Claim. □

That ends, for the moment, our general discussion of modules over rings. We shall come back to a more specialized study of these things later in the term.

§2: Categories and Functors

Now that we have a few examples under our belt I would like to spend a little time to make the fundamental definitions from category theory that we shall be using. This is only a very cursory introduction to some of the ideas of category theory. A more detailed discussion would take us too far afield since, although we will use the general ideas, we won't need (at first encounter) some of the more subtle and interesting aspects of duality from category theory.

We move to give the first (rambling) definition.

Definition: A category $\mathfrak{C}$ is:

i) a class of objects, usually denoted $Obj(\mathfrak{C})$, and usually written as $A, B, C, \ldots \in Obj(\mathfrak{C})$;

ii) for every pair of objects $A, B \in Obj(\mathfrak{C})$, we have a set, denoted $\mathfrak{C}(A, B)$. The elements of this set are called *morphisms* from A to B. Although, in general, these *morphisms* need not be functions (we'll see plenty of examples below), we use the notation:

$$\text{if } f \in \mathfrak{C}(A, B) \text{ then } f : A \to B$$

and call A the *domain* of f and B the *codomain* of f.

iii) to each triple $A, B, C \in Obj(\mathfrak{C})$ there is a *law of composition*, i.e. a function

$$\mathfrak{C}(A, B) \times \mathfrak{C}(B, C) \to \mathfrak{C}(A, C)$$

Again, following the "notation" of functions for the "not necessarily functions" of $\mathfrak{C}(-, -)$ we usually denote the composition law as "composition of morphisms" , i.e.

$$(f, g) \longrightarrow g \circ f \text{ or, simply } gf \ .$$

We require three axioms for an arrangement as above to be a category.

A1: The sets $\mathfrak{C}(A, B)$ and $\mathfrak{C}(C, D)$ are disjoint unless $A = C$ and $B = D$.

A2: The composition law is *associative*, i.e.

$$h(gf) = (hg)f \ .$$

A3: There is, for each $A \in Obj(\mathfrak{C})$, an element in $\mathfrak{C}(A, A)$, denoted 1_A, such that

$$f \circ 1_a = f \text{ for every } f \in \mathfrak{C}(A, B) \text{ and } 1_A \circ f = f \text{ for every } f \in \mathfrak{C}(C, A) \ .$$

Remarks on the Axioms:

1). Although $A1$ is a familiar idea about functions (i.e. we don't say two functions are equal unless they also have the same domain and codomain i.e. we distinguish between the function $f(x) = 5$ if we have domain all the reals and codomain either the reals, or the positive reals, or simply the number 5 !), in homological algebra the distinction is more than just pedantic.

Consider the following example from algebraic topology: let $f_1 : S^1 \to \mathbb{R}^2$ and $f_2 : S^1 \to \mathbb{R}^2 - \{(0,0)\}$. I.e. the first map embeds S^1 in the real plane with center at the origin and the second map embeds S^1 in the real punctured plane, with the center of the circle still at the origin. It's crucial to distinguish between these two functions in homotopy theory since the first is homotopic to zero while the second is not!

2). Note that composition is defined *only* if codomain $(f) =$ domain(g).

3). The element $1_A \in \mathfrak{C}(A, A)$ is uniquely determined by A (an easy exercise).

Examples: There are many examples of categories all over mathematics. In each of the examples below I will describe, first, the objects of the category and then say what the morphisms are.

1). $Obj(\mathfrak{C})$ consists of all sets. If A and B are any two sets then $\mathfrak{C}(A, B)$ consists of all the functions with domain A and codomain B.

2). This time $Obj(\mathfrak{C})$ consists of all topological spaces and if A and B are topological spaces then $\mathfrak{C}(A, B)$ consists of all continuous maps from A to B. Notice that the same set would appear more than one time among the objects of this category if it came equipped with two different topologies.

3). Let $Obj(\mathfrak{C})$ consist of all groups and if A and B are groups then $\mathfrak{C}(A, B)$ is all the group homomorphisms from A to B.

4). This time let $Obj(\mathfrak{C})$ consist of all abelian groups and $\mathfrak{C}(A, B)$ the group homomorphisms from A to B.

> The example of 4) is called a *subcategory* of the category in 3) since the objects of 4) are a subset of the objects of 3) and the morphisms between two objects of the category in 4) are a subset of the morphisms between those objects considered in the category 3).

This example is somewhat special in that whenever A and B are objects of 4), the morphisms between A and B are the same in both categories. When this is the case the smaller category is called a *full subcategory* of the larger one.

5). Fix a field F and let $Obj(\mathfrak{C})$ consist of all F-vector spaces. If V and W are two F-vector spaces then we let $\mathfrak{C}(V, W)$ be all the linear transformations from V to W.

6). Let $Obj(\mathfrak{C})$ consist of all rings, and if A and B are two rings let $\mathfrak{C}(A, B)$ be all the ring homomorphisms from A to B.

7). This time let $Obj(\mathfrak{C})$ consist of rings with identity and if A and B are such rings then $\mathfrak{C}(A, B)$ should only consist of ring homomorphisms which preserve the identity.

8). Fix a ring R and let $Obj(\mathfrak{C})$ consist of all the right (left) R-modules and let $\mathfrak{C}(A, B)$ be the R-module homomorphisms between A and B.

9) Fix a ring R and let $Obj(\mathfrak{C})$ consist of all chain complexes of R-modules. We should define the morphisms in this category.

Definition: A collection of R-module homomorphisms $\{f_n\}$, $n \in \mathbb{Z}$ is a morphism between the chain complexes $\{C_i, d_i\}$ and $\{C'_i, d'_i\}$ of R-modules if

$$\text{for each } i \in \mathbb{Z},\ f_i : C_i \to C'_i$$

and we have, for each i, the following commutative diagram

$$\begin{array}{ccc} C_i & \xrightarrow{f_i} & C'_i \\ d_i \downarrow & & \downarrow d'_i \\ C_{i-1} & \xrightarrow{f_{i-1}} & C'_{i-1} \end{array} .$$

Exercise 5A: (10) Show that this is a category.

Note that in this case the morphisms are not functions but sets of functions.

Sect.3B

10). In this example let the objects be topological spaces and the morphisms between two topological spaces to be homotopy classes of maps between them.

Using facts from algebraic topology one can see that this is indeed a category. Note also that in this example the morphisms are not functions but "homotopy classes" of functions!

11). (A strange example!)

A category with one object, denoted *, is the same thing as a monoid M, i.e. if we let $\mathfrak{C}(*,*) = M$.

In particular, any group G can be considered as a category with one object.

12). Recall that a *partially ordered set* (i.e. a poset) is a set $\mathfrak{P}$ with a reflexive, transitive and antisymmetric relation "$\leq$ ". We can regard $\mathfrak{P}$ as a category in the following way:

$$Obj(\mathfrak{P}) = \{p \mid p \in \mathfrak{P}\} \ ,$$

$$\mathfrak{P}(p,q) = \begin{cases} \emptyset & \text{unless } p \leq q; \\ \{\iota_{pq}\} & \text{one morphism from } p \text{ to } q \text{ if } p \leq q. \end{cases}$$

Some of the categories I've described above are so fundamental that they even have their own names! E.g. 1), the "mother of all categories" is called **Sets**, while 3) is called **Groups**, 4) is called **Ab** and 6) is called **Rings**. If we restrict to left R-modules, then 8) is called **R-mod**, while if we consider only right R-modules, the category is called **mod-R**. Those are among the most important names. If we start using a category a lot it is often useful to give it a name. I'll leave those baptisms until they are needed.

Further Categorical Terminology

An element $f \in \mathfrak{C}(A,B)$ is called *invertible* if there is an element $g \in \mathfrak{C}(B,A)$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

It's easy to see that if f is invertible then g is uniquely determined by f and so we may call $g = f^{-1}$. As always, we have $(f^{-1})^{-1} = f$ and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ when f and g are invertible.

If there is an invertible morphism in $\mathfrak{C}(A,B)$ we say that A is isomorphic to B in $\mathfrak{C}$. This is easily seen to be an equivalence relation on the objects of $\mathfrak{C}$, which has been given a different name in different categories.

E.g. $1-1$ correspondence in the category of sets and set mappings, homeomorphism in the category of topological spaces and continuous maps, group isomorphism in the category of groups and group homomorphisms. etc.

Sometimes a category $\mathfrak{C}$ contains a special object, often denoted $\mathbf{0}$, with the property that:

for any object $X \in Obj(\mathfrak{C})$ the sets $\mathfrak{C}(X, \mathbf{0})$ and $\mathfrak{C}(\mathbf{0}, X)$ contain a single element.

For example, the group with one element is such an object in the category of groups (or in the category of abelian groups). The module consisting only of the zero element of R is such an object in the category of left (right) R-modules. The ring with only the zero element works in the category of rings and ring homomorphisms.

An object like the one described above is called a *zero object* in the category $\mathfrak{C}$. Clearly, any two zero objects in a category $\mathfrak{C}$ are isomorphic.

If $\mathfrak{C}$ has a zero object $\mathbf{0}$, then for any $X,\ Y \in Obj(\mathfrak{C})$ we have a distinguished morphism in $\mathfrak{C}(X, Y)$ given by

$$X \longrightarrow \mathbf{0} \longrightarrow Y\ .$$

This morphism is called the *zero morphism* of $\mathfrak{C}(X, Y)$ and denoted $\mathbf{0}_{XY}$. It has the property that if $f \in \mathfrak{C}(Y, Z)$ then $f \circ \mathbf{0}_{XY} = \mathbf{0}_{XZ}$ and if $g \in \mathfrak{C}(A, X)$ then $\mathbf{0}_{XY} \circ g = \mathbf{0}_{AY}$.

Exercise 6A: (10) The map $\mathbf{0}_{XY} \in \mathfrak{C}(X, Y)$ does not depend on the zero object chosen in the category $\mathfrak{C}$.

Definitions: An object $\mathcal{I}$ in a category $\mathfrak{C}$ is called an *initial* object if $\mathfrak{C}(\mathcal{I}, X)$ consists of one element for every $X \in Obj(\mathfrak{C})$; it's called a *terminal* object (or *coinitial* object) if $\mathfrak{C}(X, \mathcal{I})$ consists of one element for every $X \in Obj(\mathfrak{C})$.

Clearly, being a zero object in the category $\mathfrak{C}$ implies that the object is both an initial and terminal object. The converse is also obviously true.

Having said that, it is interesting to note that in the category **SETS**, the empty set, $\emptyset$, is an initial object and every one point set is a terminal object. (Explain about the initial object!) It's easy to show that all initial objects in a category are isomorphic and all terminal objects are isomorphic.

Exercise 6B: (10) Prove that in the category **SETS** there is no zero object.

A morphism $f : B \to C$ in a category $\mathfrak{C}$ is called a *monomorphism* (or *mono* or *monic*) if:

$$\text{whenever } e_1, e_2 \in \mathfrak{C}(A, B) \text{ and } f \circ e_1 = f \circ e_2 \text{ then } e_1 = e_2 \ .$$

In **SETS** the monos are precisely the $1-1$ maps (in fact, that's also true in **AB** and R-**mod** and **mod**-R). In order to remember where the cancellation takes place, monos are often written $f : A \rightarrowtail B$ where the extra "$>$" is on the left – the side on which we can cancel.

Exercise 6C: (10) Let $\mathfrak{C}$ be a category in which the objects are all sets and the morphisms are all functions. Show that the monomorphsims are precisely the $1-1$ maps.

Analogously $f : B \to C$ (f a morphism in $\mathfrak{C}(B, C)$) is called an *epimorphism* (or *epic* or an *epi*) if

$$\text{whenever } e_1,\ e_2 \in \mathfrak{C}(C, D) \text{ and } e_1 \circ f = e_2 \circ f \text{ then } e_1 = e_2 \ .$$

As above, if $f : B \to C$ is an epi we write the arrow as $\twoheadrightarrow$ to remember where the cancellation can take place.

If the underlying objects of the category $\mathfrak{C}$ are sets than any "onto" morphism is an epi but the converse is not true.

Exercise 6D: *a*) (10) Show that in **Rings** the inclusion of $\mathbb{Z}$ into $\mathbb{Q}$ is an epi. More generally, whenever R is commutative with identity and S is a multiplicatively closed set, not containing 0, then the natural map $R \to S^{-1}R$ (given by $r \to [r/1]$) is an epi in **Rings**.

b) (5) Show that the inclusion of $\mathbb{Q}$ into $\mathbb{R}$ is an epi in the category of Hausdorff topological spaces and continuous maps.

Definition: Let $\mathfrak{C}$ be a category with a zero object and let $f \in \mathfrak{C}(B, C)$. A *kernel of* f is:

i) a morphism $\iota \in \mathfrak{C}(A, B)$ such that $f \circ \iota = 0$ in $\mathfrak{C}(A, C)$;

ii) which satisfies the following Universal Mapping Property:

if $g \in \mathfrak{C}(A', B)$ is any morphism such that $f \circ g = 0$ then there is a unique morphism $\chi : A' \to A$ such that:

$$\begin{tikzcd} A' \arrow[dr, "g"] \arrow[dd, "\chi"'] & & \\ & B \arrow[r, "f"] & C \\ A \arrow[ur, "\iota"] & & \end{tikzcd}$$

and the triangle commutes.

In some sense, (A, ι) is the "biggest" pair with the property that $f \circ \iota = 0$.

Claim: ι is monic.

Proof: Suppose we have $h_1,\ h_2 : D \to A$ where $\iota h_1 = \iota h_2$. Then

$$f(\iota h_1) = f(\iota h_2) = 0$$

i.e. $(f\iota)h_1 = 0$ and $(f\iota)h_2 = 0$, and we have a diagram:

$$\begin{tikzcd} A \arrow[r] & B \arrow[r, "f"] & {} \\ & A \arrow[u, "\iota"] & \\ & D \arrow[u, "h_1"] & \end{tikzcd}$$

So, we have a unique map $\chi : D \to A$ such that $\chi\iota = h_1\iota$. But $\chi = h_1 = h_2$ both work! So, $h_1 = h_2$. $\square$

It is easy to see that any two kernels of $f : B \to C$ are isomorphic.

In like manner we can define a *cokernel* of a morphism $f : B \to C$ as

i) a morphism $p : C \to D$ such that $pf = 0$ and

ii) satisfying the following Universal Mapping Property:

if $g : C \to D'$ is any morphism such that $gf = 0$ then there is a unique morphism $\chi : D \to D'$ such that the following diagram commutes:

$$\begin{tikzcd} & & D \arrow[dd, "\chi"] \\ B \arrow[r, "f"] & C \arrow[ur, "p"] \arrow[dr, "g"'] & \\ & & D' \end{tikzcd}$$

In a sense, the pair (D, p) is the "smallest" with the property that $pf = 0$.

It's easy to show (using the UMP) that a coker is an epi and any two cokernels of $f : B \to C$ are isomorphic.

Continuing, we define a *product* of objects $\{C_i\}_{i\in\mathcal{I}}$ in the category $\mathfrak{C}$, as an object (denoted $\Pi_{i\in\mathcal{I}}C_i$) together with morphisms π_j, where

$$\pi_j : \Pi_{i\in\mathcal{I}}C_i \to C_j$$

satisfying the Universal Mapping property we described earlier in the lectures.

Likewise, a *coproduct* of objects $\{C_i\}_{i\in\mathcal{I}}$ in the category $\mathfrak{C}$, is an object (denoted $\coprod C_i$) together with a family of morphisms

$$\iota_j : C_j \to \coprod C_i$$

satisfying the UMP we saw earlier. (Note: sometimes the coproduct is called a "sum".)

We've seen that the category R-**mod** has products and coproducts for any family of objects.

Examples: *i*) If $\mathfrak{C}$ is the category of commutative rings with identity, then $\mathbb{Z}$ is an initial object. The zero ring (which has one element $0 = 1$) is a terminal object. There is no zero object.

It's easy to see that $R_1 \times R_2$ (with pairwise operations) is a product in the category of **RINGS**, but it is **NOT** a coproduct! (There is a coproduct in this category, we'll see later that it is $R_1 \otimes_{\mathbb{Z}} R_2$.) There are no kernels or cokernels in this category since there is no zero object.

ii) Let $\mathfrak{C}$ be **GROUPS**. The group $\{e\}$ is a **0**-object in $\mathfrak{C}$. The cartesian product $G_1 \times G_2$ is easily seen to be a product but not a coproduct in this category. There is a coproduct, but it is not so easy to describe.

Kernels are easy to describe. On the other hand, if $f : G_1 \to G_2$ then $coker(f)$ is

$$\pi : G_2 \to G_2/(\text{normal subgroup generated by } Im(f)).$$

Notice that the inclusion of a subgroup into a group is a mono, but it is not a kernel unless the subgroup is normal in the group. (I suggest you verify that a non-normal subgroup of a group is NOT a kernel in the category **GROUPS**.)

Sect.4A

Special Kinds of Categories

An *additive category* $\mathfrak{C}$ is a category such that:

i) There is a zero object in the category; and

ii) For every $A,\ B \in Obj(\mathfrak{C})$ we have that $\mathfrak{C}(A,B)$ is an abelian group and, moreover, the composition

$$\mathfrak{C}(A,B) \times \mathfrak{C}(B,C) \to \mathfrak{C}(A,C)$$

is bilinear, i.e. if $f,\ g \in \mathfrak{C}(A,B)$ and $h \in \mathfrak{C}(B,C)$ then

$$h \circ (f+g) = h \circ f + h \circ g$$

(and the bilinearity on the other side as well!)

iii) If $A,\ B \in Obj(\mathfrak{C})$ then $A \prod B$ exists in $\mathfrak{C}$.

It is easy to see that **Ab** is an additive category, as is R-**mod** and **mod**-R. The apparent asymmetry in the definition (which asks only for finite products, but not finite coproducts) is more apparent than real. It's not hard to prove that in an additive category the two notions agree.

We now come to the most important kind of category that we will be considering this year, the abelian category. We give the complete definition.

Definition: An *abelian category* is a category $\mathfrak{C}$ which is an additive category and which satisfies, in addition, the following properties:

i). Every morphism in $\mathfrak{C}$ has a kernel and a cokernel.

ii). Every monic is the kernel of its cokernel.

iii). Every epi is the cokernel of its kernel.

Proposition: The category **mod**-R is an abelian category (and, we'll see, is the prototype of an abelian category).

Proof: To see i). Let $f : A \to B$ be a homomorphism of R-modules and set

$$K(A) = \{a \in A \mid f(a) = 0\}$$

and let $\iota : K(f) \to A$ the inclusion of $K(f)$ into A.

Claim: The pair $(K(f), \iota)$ is a kernel of f.

Proof: Suppose we have a diagram

$$\begin{array}{ccccc} K(f) & \xrightarrow{\iota} & A & \xrightarrow{f} & B \\ & \nearrow g & & & \\ C & & & & \end{array}$$

and suppose that $fg = 0$.

If $c \in C$, then $g(c) \in A$ and $f(g(c)) = 0$ and so $g(c) \in K(f)$. Let $\chi : C \to K(f)$ be defined by $\chi(c) = g(c)$. Then χ is the required map which makes the diagram commute and is easily verified to be an R-module homomorphism. $\square$

Let $Im(f) \subseteq B$. Then $Im(f)$ is a submodule of B and we can define

$$\pi : B \to B/Im(f) \ .$$

Claim: The pair $(\pi, B/Im(f))$ is a cokernel of f.

Proof: Suppose we have a diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{\pi} & B/Im(f) \\ & & & g \searrow & \\ & & & & C \end{array}$$

where $gf = 0$. We want a unique map $B/Im(f) \to C$ making the diagram commute.

Define $\chi(\overline{b}) = g(b)$. As soon as one shows this is well-defined it is easily seen to be an R-module homomorphism and is the required map. $\square$

As for ii), we must show

a) that every mono is the kernel of its cokernel.

So, let $f : A \to B$ be a mono. Then f is $1-1$ (as we have seen!) We have just seen that the cokernel of f is the natural map $\pi : B \to B/Im(f)$. So, we must show that f is the kernel of π.

So, et $g : C \to B$ be such that $\pi g = 0$. We want to find a unique map $\chi : C \to A$ making the appropriate diagram commute.

If $c \in C$ then $g(c) \in Ker(\pi) = Im(f) \simeq A$. Thus, $\chi(c) = f^{-1} \circ g(a)$ makes sense and is the only thing that will work.

b) Every epi is the cokernel of its kernel.

I.e. given an epi $p : A \twoheadrightarrow B$ and $\iota : Ker(p) \to A$ we must show that p is the cokernel of ι.

I.e. given $g : A \to C$, we want to find a unique map $\chi : B \to C$ making the appropriate diagram commute.

Let $b \in B$, then $b = p(a), a \in A$ (I am using the fact that epis are onto in the category R-**mod** !!!). Define $\chi(b) = g(a)$. This is the only map making the diagram commute, so it's enough to show it is well defined, i.e. if $b \in B$ and $b = p(a_1) = p(a_2)$ then $g(a_1) = g(a_2)$.

But, $p(a_1) = p(a_2) \Rightarrow p(a_1 - a_2) = 0 \Rightarrow a_1 - a_2 \in Ker(p) \Rightarrow (g\iota)(a_1 - a_2) = 0 \Rightarrow g(\iota(a_1)) = g(\iota(a_2))$. Since $\iota a = a$ we are done.

Exercise 7A: *a*) (10) Prove that in an abelian category a morphism which is both mono and epi is an isomorphism.

b) (5) Show, with an example, that this is not necessarily true in an arbitrary category.

Having talked about kernels and cokernels it is possible to speak about short exact sequences in an abelian category.

Exercise 7B: *a*) (10) If $\mathfrak{A}$ is an abelian category show that $Ch(\mathfrak{A})$, the chain complexes over $\mathfrak{A}$ is also an abelian category.

b) (5) Moreover, show that

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

is a short exact sequence of chain complexes if and only if

$$0 \to A_n \to B_n \to C_n \to 0$$

is a short exact sequence in $\mathfrak{A}$. (be careful! the objects here may not be sets!)

In the light of this exercise, we can thus talk about chain complexes of chain complexes of chain complexes of .....

The *next* step of this process is interesting.

Definition: Let $\mathfrak{A}$ be an abelian category. A *double complex* or *bicomplex* over $\mathfrak{A}$ is a family $\{C_{p,q}\}$ where $C_{p,q} \in Obj(\mathfrak{A})$ such that:

$$d_{p,q} : C_{p,q} \to C_{p-1,q} \text{ and } d'_{p,q} : C_{p,q} \to C_{p,q-1}$$

and $d^2 = 0$ and $(d')^2 = 0$.

These double complexes are usually pictured as follows:

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & C_{p+1,q+1} & \stackrel{d}{\to} & C_{p,q+1} & \stackrel{d}{\to} & C_{p-1,q+1} & \stackrel{d}{\to} & \cdots \\
 & & d' \downarrow & & d' \downarrow & & d' \downarrow & & \\
\cdots & \to & C_{p+1,q} & \stackrel{d}{\to} & C_{p,q} & \stackrel{d}{\to} & C_{p-1,q} & \stackrel{d}{\to} & \cdots \\
 & & d' \downarrow & & d' \downarrow & & d' \downarrow & & \\
\cdots & \to & C_{p+1,q-1} & \stackrel{d}{\to} & C_{p,q-1} & \stackrel{d}{\to} & C_{p-1,q-1} & \stackrel{d}{\to} & \cdots \\
 & & d' \downarrow & & d' \downarrow & & d' \downarrow & &
\end{array}$$

where $d_{p,q-1} \circ d'_{p,q} + d'_{p-1,q} \circ d_{p,q} = 0$, i.e. each square is anti-commutative. (Note that this is not quite what the morphisms in $Ch(Ch(\mathfrak{A}))$ would be like, but it would be OK if we changed the morphisms in every other column by a minus sign!)

The anticommutativity condition is useful because we can make two new complexes out of the double complex $\mathcal{D}$ above, called **Total Complexes**, as follows:

$$(Tot^{\prod}\mathcal{D})_n = \prod_{p+q=n} C_{p,q} \quad , \quad (Tot^{\coprod}(\mathcal{D}))_n = \coprod_{p+q=n} C_{p,q}$$

where $\tilde{d} = d + d'$ defines a morphism (in each case) and $\tilde{d}^2 = 0$ (check this out!).

Exercise 7C: (10) A double complex is *bounded* if it has only finitely many non-zero terms along each diagonal $p + q = n$ (e.g. if it is concentrated in the first quadrant.)

Prove that $Tot(\mathcal{C})$ is acyclic (i.e. has zero homology modules) when $\mathcal{C}$ is a bounded double complex with exact rows (or exact columns). You needn't do this in an arbitrary abelian category – it is enough for you to do this for R-modules.

FUNCTORS

So far, we have only been discussing one single category at a time, but, that is like studying only one group at a time. Just as we have seen that functions are very important in the study of various algebraic and geometric objects, we need to have the appropriate analogue in the study of categories. The appropriate notion appears to be that of a *functor*.

Definition: Let $\mathfrak{C}$ and $\mathfrak{D}$ be two categories. A (covariant) *functor* from $\mathfrak{C}$ to $\mathfrak{D}$, written

$$\mathcal{F} : \mathfrak{C} \longrightarrow \mathfrak{D}$$

is a rule which assigns to each $A \in Obj(\mathfrak{C})$ an element $\mathcal{F}(A) \in Obj(\mathfrak{D})$ and to every morphism $f \in \mathfrak{C}(A, B)$ a morphism $\mathcal{F}(f) \in \mathfrak{D}(\mathcal{F}(A), \mathcal{F}(B))$ such that

i). $\mathcal{F}(id_A) = id_{\mathcal{F}(A)}$, and

ii). $\mathcal{F}(gf) = \mathcal{F}(g)\mathcal{F}(f)$.

Examples:

1). If R is any ring and M is in R-**mod**, then we have shown that $Hom_R(M, -)$ can be viewed as a functor from the category R-**mod** to the category **Ab**.

2). If R is a commutative ring and M is a fixed R-module then $Hom_R(M, -)$ is a functor from the category R-**mod** to itself.

3). The functor $\mathcal{F} : R$-**mod** $\to$ **Ab** which $\underline{forgets}$ that an R-module is an R-module and just remembers that it is an abelian group, is an example of, what is often called, a "*forgetful*" functor.

4). If A is an abelian group and we let

$$T(A) = \{g \in A \mid ng = 0 \text{ for some } n \in \mathbb{Z}\}$$

(called the *torsion subgroup* of A), then the assignment $T : A \to T(A)$ describes a functor from the category **Ab** to itself.

Remark: If, in the definition of a functor, we assume (instead) that

$$\mathcal{F}(f) \in \mathfrak{D}(\mathcal{F}(B), \mathcal{F}(A)) \text{ and also that } \mathcal{F}(gf) = \mathcal{F}(f)\mathcal{F}(g) \ ,$$

then $\mathcal{F}$ is called a *contravariant* functor.

The functors $Hom_R(-, N)$ (where R is a commutative ring and N is a fixed R-module) are contravariant functors from R-**mod** to itself.

Special Functors:

In explaining some additional functor possibilities, we shall assume in what follows, that $\mathcal{F} : \mathfrak{C} \longrightarrow \mathfrak{D}$ is a functor.

i). $\mathcal{F}$ is called *faithful* if, for all $A, \ B, \ \in Obj(\mathfrak{C})$ then the map

$$\mathfrak{C}(A, B) \longrightarrow \mathfrak{D}(\mathcal{F}(A), \mathcal{F}(B))$$

is a $1 - 1$ map of sets.

The forgetful functor above is an example of such a functor.

ii). $\mathcal{F}$ is call *full* if, for all $A, \ B, \ \in Obj(\mathfrak{C})$, the map

$$\mathfrak{C}(A, B) \longrightarrow \mathfrak{D}(\mathcal{F}(A), \mathcal{F}(B))$$

is an onto mapping of sets.

If $\mathfrak{A}$ is the category of finitely generated abelian groups and $T : \mathfrak{A} \to \mathfrak{A}$ is the torsion functor described above, then T is a full (but not faithful) functor.

iii). A functor which is both faithful and full is called *fully faithful.* The inclusion of a full subcategory of a category into that category is a typical fully faithful functor.

Sect.4B

iv). If $\mathfrak{C}$ and $\mathfrak{D}$ are both additive categories, then $\mathcal{F}$ is called an *additive* functor if the assignment

$$\mathcal{F} : \mathfrak{C}(A, B) \to \mathfrak{D}(\mathcal{F}(A), \mathcal{F}(B))$$

is a group homomorphism for every $A, \ B \in Obj(\mathfrak{C})$.

v). If now $\mathfrak{C}$ and $\mathfrak{D}$ are abelian categories we can further speak of *left exact*, *right exact* and *exact* functors. We've seen these definitions earlier, done informally, and I leave them to you to formulate exactly!!!

One of the most important theorems of Category Theory talks about (what are called) "small " categories (i.e. categories for which the collection of objects is a set). We will not give the proof here.

Freyd-Mitchell Embedding Theorem: If $\mathfrak{A}$ is a small abelian category, then there is a ring R and an exact, fully faithful functor from $\mathfrak{A}$ into R-**mod** which embeds $\mathfrak{A}$ as a full subcategory of R-**mod**.

(This means that a great many arguments in a general abelian category can be made by using the same argument in the category R-**mod**).

Natural Transformations of Functors

Definitions: Let $\mathcal{F}$ and $\mathcal{G}$ be two functors from $\mathfrak{C}$ to $\mathfrak{D}$.

A *natural transformation* $\eta : \mathcal{F} \to \mathcal{G}$ is a rule which associates to every object $A \in Obj(\mathfrak{C})$, a **fixed** morphism $\eta_A : \mathcal{F}(A) \to \mathcal{G}(A)$ in $\mathfrak{D}$ such that, for every morphism $f \in \mathfrak{C}(A, B)$ in $\mathfrak{C}$, we have a commutative diagram

$$\begin{array}{ccccc} A & \quad & \mathcal{F}(A) & \xrightarrow{\eta_A} & \mathcal{G}(A) \\ f \downarrow & & \mathcal{F}(f)\downarrow & & \downarrow \mathcal{G}(f) \\ B & & \mathcal{F}(B) & \xrightarrow{\eta_B} & \mathcal{G}(B) \end{array}$$

If each η_A is an isomorphism we say that η is a *natural isomorphism* and write $\mathcal{F} \simeq \mathcal{G}$.

A functor $\mathcal{F} : \mathfrak{C} \to \mathfrak{D}$ is called an *equivalence of categories* if there is a functor $\mathcal{G} : \mathfrak{D} \to \mathfrak{C}$ such that

$$\mathcal{G} \circ \mathcal{F} \simeq id_{\mathfrak{C}} \text{ and } \mathcal{F} \circ \mathcal{G} \simeq id_{\mathfrak{D}} \ .$$

Example: A simple example of an equivalence of categories is the following: let $\widetilde{V_k}$ = category of <u>*based*</u> vector spaces, i.e. the objects are vector spaces over k, each with a fixed basis. Then the morphisms are appropriately sized matrices.

Let V_k be the category of all vector spaces and vector space linear transformations.

The claim is that these categories are equivalent.

To see this, let $\mathcal{F}$ be the forgetful functor which "forgets" the chosen basis. Clearly, many objects get sent to the same thing by this functor. Let $\mathcal{G}$ be the functor which picks a basis for each vector space. Then it is easy to see that $\mathcal{F} \circ \mathcal{G} = id$ and clearly $\mathcal{G} \circ \mathcal{F} \neq id$.

Claim: $\mathcal{G} \circ \mathcal{F} \simeq id_{\widetilde{V_k}}$

Proof: We want to find, for each $(V, \beta) \in Obj(\widetilde{V_k})$, where β is a basis for the vector space V, an isomorphism between

$$\mathcal{GF}(V, \beta) \text{ and } (V, \beta)$$

making the appropriate diagram commute. i.e. we want:

$$\begin{array}{ccc} (V, \beta') = \mathcal{GF}(V, \beta) & \xrightarrow{[id]^{\beta}_{\beta'}} & (V, \beta) \\ [f]^{\gamma'}_{\beta'} \downarrow & & [f]^{\gamma}_{\beta} \\ \mathcal{GF}(W, \gamma) = (W, \gamma') & \xrightarrow{[id]^{\gamma}_{\gamma'}} & (W, \gamma) \end{array}$$

But, this is standard from linear algebra.

Tensor Products

Let R be a commutative ring and let $E_1, \ldots, E_n,\ F$ be R-modules. A function

$$f : E_1 \times \ldots \times E_n \longrightarrow F$$

is called *multilinear* if:

given i, $1 \leq i \leq n$ and, for each $j \neq i$ any elements $x_j \in E_j$, then the function

$$f(x_1, \ldots, x_{i-1}, -, x_{i+1}, \ldots, x_n) : E_i \longrightarrow F$$

is an R-module homomorphism.

Example: If V is a vector space over the field k (with a fixed basis) and $dim(V) = n$, then if we let $V_i = V$ for $i = 1, \ldots, n$, then

$$det : V_1 \times \cdots \times V_n \longrightarrow k$$

defined by $det(v_1, \ldots, v_n) =$ the determinant of the $n \times n$ matrix whose columns are the coordinates of the v_i with respect to the chosen basis, is a well-known multilinear form.

We denote by $M(E_1, \ldots, E_n; F)$ the set of all such multilinear maps. This set can be made into an R-module by using the R-module structure on F, via

$$(rf)(x_1, \ldots, x_n) = r(f(x_1, \ldots, x_n))$$

and

$$(f+g)(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) + g(x_1, \ldots, x_n) \ .$$

(Note: As earlier, this may not work if R is not commutative.)

We define the tensor product of $E_1, E_2, \cdots, E_n$ over R as follows:

let M be the free R-module on the set $E_1 \times \cdots \times E_n$. We will denote the basis element in M associated to the element $(e_1, \ldots, e_n)$ by $1_{(e_1,\ldots,e_n)}$ and let

$$\phi : (e_1, \ldots, e_n) \to 1_{(e_1,\ldots,e_n)}$$

be a function from $E_1 \times \cdots \times E_n$ to M.

Now let N be the submodule of M generated by all the elements of the form:

$$1_{(x_1,\ldots,x_{i-1},x_i+x_i',x_{i+1},\ldots,x_n)} - 1_{(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n)} - 1_{(x_1,\ldots,x_{i-1},x_i',x_{i+1},\ldots,x_n)} \ ,$$

and

$$1_{(x_1,\ldots,x_{i-1},ax_i,x_{i+1},\ldots,x_n)} - a1_{(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n)} \ a \in R \ .$$

If we now consider $\eta \circ \phi : E_1 \times \cdots \times E_n \to M \xrightarrow{\eta} M/N$, then we see that we have factored out just enough to make $\eta \circ \phi$ a multilinear map.

Definition: We define $M/N = E_1 \otimes_R E_2 \otimes_R \cdots \otimes_R E_n$ and call this the tensor product of $E_1, \ldots, E_n$ over R. It is an R-module.

If we let $\chi = \eta \circ \phi$, then $\chi(x_1, \ldots, x_n)$ will be denoted by $x_1 \otimes x_2 \otimes \cdots \otimes x_n$. Note that not every element of the tensor product is of this form. Rather it is a sum of elements of this form!

Theorem: The multilinear map

$$\chi : E_1 \times \cdots E_n \longrightarrow E_1 \otimes_R E_2 \otimes_R \cdots \otimes_R E_n$$

satisfies the following UMP:

If f is any multilinear map out of $E_1 \times \cdots E_n$ then there is a *unique* **linear** map f_* making the following diagram commute:

$$\begin{tikzcd}
E_1 \times \cdots E_n \arrow[r, "f"] \arrow[d, "\chi"'] & F \\
E_1 \otimes_R E_2 \otimes_R \cdots \otimes_R E_n \arrow[ur, "f_*"'] &
\end{tikzcd}$$

Proof: We have a diagram

$$\begin{tikzcd}
E_1 \times \cdots E_n \arrow[r, "f"] \arrow[d, "\phi"'] & F \\
M \arrow[ur, "g"'] \arrow[d, "\eta"'] & \\
M/N = E_1 \otimes_R E_2 \otimes_R \cdots \otimes_R E_n \arrow[uur, "g"'] &
\end{tikzcd}$$

Let $g(1_{(x_1,\ldots,x_n)}) = f(x_1,\ldots,x_n)$. Then $g \circ \phi = f$ as set functions. We can extend that to a unique homomorphism of R-modules, since M is free, still called g.

Since f is multilinear, this implies that g vanishes on N, so g passes to an R-module homomorphism, also denoted by g, which makes the diagram commute. The function g is unique, as constructed. $\square$

It is not too surprising that the UMP for the tensor product is one of the fundamental features of the object. It "converts" multilinear maps into linear ones!

Simple Properties of the Tensor Product

Proposition: Let R be a commutative ring. There is a natural isomorphism of R-modules,

$$f : R \otimes_R M \to M$$

given by $f(r \otimes m) = rm$.

Proof: Consider $g : R \times M \to M$ by $g(r,m) = rm$. Since M is an R-module, g is multilinear.

By the UMP we have a map g_* making the following diagram commute:

$$\begin{array}{ccc} R \times M & \xrightarrow{g} & M \\ \downarrow \chi & \nearrow g_* & \\ R \otimes_R M & & \end{array}$$

where $g_*(r \otimes m) = rm$.

Clearly g_* is onto, so it suffices to show that g_* is $1-1$.

A typical element x of $R \otimes_R M$ is $x = \sum_{i=1}^n a_i(r_i \otimes m_i)$ where $a_i \in R$, $r_i \in R$ and $m_i \in M$. By the standard properties of the tensor product we have:

$$a_i(r_i \otimes m_i) = (a_i r_i) \otimes m_i = 1 \otimes (a_i r_i m_i) = 1 \otimes m_i' \ .$$

So,

$$x = \sum_{i=1}^n 1 \otimes m_i' = 1 \otimes (\sum_{i=1}^m m_i') = 1 \otimes m$$

for a suitable $m \in M$.

Thus, $g_*(x) = g_*(1 \otimes m) = 1m = m$, so $g_*(x) = 0$ implies that $m = 0$, i.e. $x = 1 \otimes 0$. But then $x = 0$ and so g_* is $1-1$ as we wanted to show. $\square$

We now want to see how the tensor product behaves with respect to homomorphisms.

Let $f : M_1 \to M_2$ and let $g : N_1 \to N_2$ be R-module homomorphisms.

Claim: There is an R-module homomorphism denoted $f \otimes g$ defined as follows

$$f \otimes g : M_1 \otimes N_1 \to M_2 \otimes N_2 \text{ by } (f \otimes g)(m_1 \otimes n_1) = f(m_1) \otimes g(n_1) \ .$$

Proof: The method should now be becoming clear! Consider the map $(f,g) : M_1 \times N_1 \to M_2 \otimes N_2$ given by $(f,g)(m_1, n_1) = f(m_1) \otimes g(n_1)$.

If we prove that (f, g) is a multilinear map, then the induced map on the tensor product is exactly what we are looking for. $\square$

Note: If f and g are onto, so is $f \otimes g$.

Exercise 8A: $i)$ (10) Show that $E_1 \otimes_R E_2 \simeq E_2 \otimes_R E_1$.

$ii)$ (10) Show that

$$(E_1 \otimes E_2) \otimes E_3 \simeq E_1 \otimes E_2 \otimes E_3 \simeq E_1 \otimes (E_2 \otimes E_3) \ .$$