Part 2 of Tony Geramita's homological algebra notes, Fall 94.

sect5B.tex

We shall continue listing some of the simple properties of the tensor product.

Proposition: Let $M = \oplus_{i\in\mathcal{I}} M_i$ and P be R-modules. Then

$$(\oplus M_i) \otimes P \simeq \oplus(M_i \otimes P) \ ,$$

where the isomorphism is given by:

$$(\ldots, m_i, \ldots) \otimes p \to (\ldots, m_i \otimes p, \ldots) \ .$$

Proof: For simplicity in notation I shall only do this for two summand, i.e. I'll prove that

$$(M \oplus N) \otimes P \simeq (M \otimes P) \oplus (N \otimes P)$$

with isomorphism given by

$$(a, b) \otimes p \to (a \otimes p, b \otimes p) \ .$$

Consider the function:

$$(M \oplus N) \times P \to (M \otimes P) \oplus (N \otimes P)$$

given by

$$((a, b), p) \to (a \otimes p, b \otimes p) \ .$$

This is easily seen to be bilinear, so there is a unique map

$$(M \oplus N) \otimes P \to (M \otimes P) \oplus (N \otimes P)$$

given by

$$(a, b) \otimes p \to (a \otimes p, b \otimes p) \ .$$

Now, we have a map

$$M \times P \to (M \oplus N) \otimes P$$

given by : $(a,p) \to ((a,0) \otimes p)$. This is easily seen to be multilinear, so there is a unique linear map from $M \otimes P$ to $(M \oplus N) \otimes P$ which extends the one above. Similarly with $N \otimes P$. So we have, by the Universal Mapping Property of the Direct Sum, a map

$$(M \otimes P) \oplus (N \otimes P) \to (M \oplus N) \otimes P$$

given by

$$((m \otimes p), (n \otimes p')) \longrightarrow (m,0) \otimes p + (0,n) \otimes p' \ .$$

If we take the compositions of these two maps, first as:

$$(m \otimes p, n \otimes p') \to (m,0) \otimes p + (0,n) \otimes p' \to (m \otimes p, 0 \otimes p) + (0 \otimes p', n \otimes p') = (m \otimes p, n \otimes p')$$

which is the identity, (as is the other composition, which I leave for the reader to verify). $\square$

This proposition has a very useful consequence.

Corollary: If M and N are finitely generated R-modules, so is $M \otimes_R N$.

Proof: We have a surjection $\phi_1 : F_1 \twoheadrightarrow M$, where $F_1 \simeq \oplus_{i=1}^n R_i$ (where $R_i \simeq R$). Similarly, we have $\phi_2 : F_2 \twoheadrightarrow N$ where $F_2 \simeq \oplus_{i=1}^t R$.

Then, as we saw earlier, $\phi_1 \otimes \phi_2 : F_1 \otimes F_2 \twoheadrightarrow M \otimes N$.

But, from the previous proposition we have:

$$F_1 \otimes F_2 \simeq R^{nt}$$

and so $M \otimes N$ is also finitely generated. $\square$

Note: It is easy to see, from this Corollary, that if

$$M = < m_1, \ldots, m_n > \text{ and } N = < n_1, \ldots, n_t >$$

then

$$M \otimes_R N = < m_i \otimes n_j | 1 \leq i \leq n, 1 \leq j \leq t > \ .$$

An extremely useful fact about tensor products is the following:

Proposition: If M is an R-module and I is an ideal in R, then

$$M \otimes_R R/I \simeq M/IM$$

as R-modules.

Proof: There is a function $M \times R/I \to M/IM$ given by

$$(m, r+I) \to rm + IM \ .$$

This is easily seen to be a bilinear map, so, by the UMP for the tensor product, there is a unique linear map

$$\chi : M \otimes_R R/I \to M/IM$$

given by $m \otimes \overline{r} \to \overline{rm}$.

We would like to show that this map χ is an isomorphism. It is clearly surjective. Let $x \in M \otimes R/I$. Then $x = \sum m_i \otimes \overline{r_i}$. Since $m_i \otimes \overline{r_i} = r_i m_i \otimes \overline{1}$ we get that

$$x = \sum r_i m \otimes \overline{1} = (\sum r_i m_i) \otimes \overline{1} = m \otimes \overline{1} \text{ for some } m \in M \ .$$

So, suppose that $x \in Ker(\chi)$, i.e. $\chi(x) = 0$. Write $x = m \otimes \overline{1}$, then $\chi(x) = m + IM = 0$ implies that $m \in IM$, i.e. $m = \sum j_i m_i$ with $j_i \in I$. In that case,

$$m \otimes \overline{1} = (\sum j_i m_i) \otimes \overline{1} = \sum m_i \otimes \overline{j_i} \ .$$

But, $\overline{j_i} = 0$ and so $x = 0$, as we wanted to show. $\square$

Exercise 9A: 1). (10) Prove that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/d\mathbb{Z}$, $d = gcd(m,n)$. In particular, if $gcd(m,n) = 1$, then the tensor product above is 0.

2) (10) Let $R = k[x,y]$ and let $I = (x,y)$. Is $x \otimes y - y \otimes x$ zero or non-zero in $I \otimes_R I$? Explain! What about as an element of $R \otimes_R R$?

Now, let A, B, C be R-modules and let

$$\alpha : A \times B \longrightarrow C \text{ be a bilinear form.}$$

For a fixed $a \in A$, the map $\alpha(a, -) : B \to C$ is an R-linear map. Thus, the assignment

$$a \to \alpha(a, -)$$

is a map from $A \to Hom_R(B, C)$.

Claim: This assignment is an R-module homomorphism.
Proof: Left to the reader.

Conversely, if $\phi : A \to Hom_R(B, C)$ is an R-module homomorphism, then ϕ induces a bilinear form

$$\alpha_\phi : A \times B \to C$$

described by

$$\alpha_\phi(a, b) = [\phi(a)](b) \ .$$

In fact, the correspondences we gave above are inverse to each other (check this out!) and so we have a $1-1$ correspondence

$$Hom_R(A, Hom_R(B, C)) \leftrightarrow M(A, B; C) \ .$$

On the other hand, we have seen that there is a $1-1$ correspondence

$$M(A, B; C) \leftrightarrow Hom_R(A \otimes B, C)$$

Proposition: The composition of these $1-1$ correspondences, i.e.

$$Hom_R(A, Hom_R(B, C)) \leftrightarrow M(A, B; C) \leftrightarrow Hom_R(A \otimes B, C)$$

is an isomorphism of R-modules.

Proof: Once we show that the composition is an R-module homomorphism it will automatically be an isomorphism.

Thus, it remains to give a good description of the composition.

Pick ϕ,

$$\phi : A \to Hom_R(B, C) \leftrightarrow \alpha_\phi(a, b) = [\phi(a)](b) \leftrightarrow \alpha_\phi(a \otimes b) = (\phi(a))(b)$$

It is an easy exercise to show that $\chi : \phi \to \alpha_\phi$ is an R-module homomorphism. I'll leave that as an unassigned assignment.

Functorial Properties of the Tensor Product

Let R be a commutative ring and let M be a fixed R-module. We can consider $-\otimes_R M$ as a functor from R-modules to R-modules by

$$N \longrightarrow N \otimes_R M$$

on the objects, and if $f \in Hom_R(N_1, N_2)$ then $f \otimes id : N_1 \otimes_R M \to N_2 \otimes_R M$. So, $-\otimes_R M$ is a covariant functor.

Proposition: For R, M as above, $-\otimes_R M$ is a right exact functor, i.e.

If

$$0 \to N_1 \xrightarrow{\phi_1} N_2 \xrightarrow{\phi_2} N_3 \to 0$$

is an exact sequence of R-modules, then

$$N_1 \otimes_R M \xrightarrow{\phi_1 \otimes id} N_2 \otimes_R M \xrightarrow{\phi_2 \otimes id} N_3 \otimes_R M \to 0$$

is also exact.

Proof: Given

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

we get

$$0 \to Hom_R(N_3, P) \to Hom_R(N_2, P) \to Hom_R(N_1, P)$$

is exact, and so

$$0 \to Hom_R(M, Hom_R(N_3, P)) \to Hom_R(M, Hom_R(N_2, P))$$

$$\to Hom_R(M, Hom_R(N_1, P))$$

is exact also.

Using the isomorphism above, this gives

$$0 \to Hom_R(M \otimes N_3, P) \to Hom_R(M \otimes N_2, P) \to Hom_R(M \otimes N_1, P)$$

is exact. But this, in turn, implies that

$$M \otimes N_1 \to M \otimes N_2 \to M \otimes N_3 \to 0$$

is exact, as we wanted to show. $\square$

Example: The functor $- \otimes_R M$ is not, in general, exact. Consider the sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$$

and let $M = \mathbb{Z}/2\mathbb{Z}$.

The sequence

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{2 \otimes id} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$

is not exact since $x \otimes \overline{y}$ has image $2x \otimes \overline{y} = x \otimes 2\overline{y} = x \otimes 0 = 0$, i.e. $2 \otimes id$ is the zero map. Since

$$\Big(\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})\Big) \simeq \mathbb{Z}/2\mathbb{Z} \neq 0$$

we are done.

Definition: If the functor $- \otimes_R M$ is an exact functor we say that M is a *flat* R-module.

Proposition: The ring R is a flat R-module.

Proof: It clearly suffices to prove that:

$$\text{if } 0 \to A \xrightarrow{f} B \text{ is exact, then}$$

$$0 \to A \otimes_R R \xrightarrow{f \otimes id} B \otimes_R R \text{ is exact.}$$

But, we have

$$\begin{array}{ccccc} 0 & \to & A \otimes_R R & \xrightarrow{f \otimes id} & B \otimes_R R \\ & & \simeq\downarrow & & \simeq\downarrow \\ & & A & \xrightarrow{f} & B \end{array}$$

as a commutative diagram, and that is enough to finish the proof. $\square$

Exercise 9B: Let M_i $(i \in I)$ be a family of R-modules, and let $M = \oplus M_i$. Then

$$M \text{ is } R\text{-flat} \;\Leftrightarrow\; \text{each } M_i \text{ is } R\text{-flat.}$$

It follows from Exercise (B) that projective R-modules are all flat. It is also a simple matter to verify that if M and N are R-flat then so is $M \otimes_R N$.

Sect.6A

Aside: I seem to have left out part of the following Proposition!!

Proposition:

1). Let

$$M_1 \xrightarrow{\phi_1} M_2 \xrightarrow{\phi_2} M_3 \to 0 \qquad (\dagger)$$

be a sequence of R-modules.

The sequence $(\dagger)$ is exact $\Leftrightarrow$ for every R-module N, we have

$$0 \to Hom_R(M_3, N) \xrightarrow{\phi_2^*} Hom_R(M_2, N) \xrightarrow{\phi_1^*} Hom_R(M_1, N)$$

is exact.

2). Let

$$0 \to N_1 \xrightarrow{\phi_1} N_2 \xrightarrow{\phi_2} N_3 \qquad (\dagger\dagger)$$

be a sequence of R-modules.

The sequence $(\dagger\dagger)$ is exact $\Leftrightarrow$ for every R-module M, we have

$$0 \to Hom_R(M, N_1) \xrightarrow{(\phi_1)_*} Hom_R(M, N_2) \xrightarrow{(\phi_2)_*} Hom_R(M, N_3)$$

is exact.

Proof: We've shown the "$\Rightarrow$" parts of 1) and 2) earlier in the course. I'll just do 2), $\Leftarrow$ and leave the rest as an exercise.

Now $(\phi_2)_*(\phi_1)_* = 0$ by assumption, and so, for any $f \in Hom_R(M, N_1)$ we must have $(\phi_2)_*(\phi_1)_*(f) = 0$, i.e. $\phi_2 \circ \phi_1 \circ f = 0$. But, if we take $M = N_1$ and $f = id$, we see that this implies that $\phi_2 \circ \phi_1 = 0$, which is what we wanted to show.

Now suppose that ϕ_1 were not $1-1$, i.e. there is an R-module K, $K \neq 0$, $K = Ker(\phi_1)$. Let $M = K$ and let $i : K \to N_1$ be the inclusion of K into N_1. By assumption

$$0 \to Hom_R(K, N_1) \xrightarrow{(\phi_1)_*} Hom_R(K, N_2)$$

is exact, i.e. $(\phi_1)_*$ is $1-1$. But, clearly $\phi_i \circ i = 0$, i.e. $(\phi_1)_*(i) = 0$, so i is the zero map. That contradiction establishes this part of the proposition.

So, it remains only to show that $Im(\phi_1) \supseteq Ker(\phi_2)$. Recall that we are given that $Im((\phi_1)_*) = Ker((\phi_2)_*)$.

Let $K = Ker(\phi_2)$ and let $i : K \to N_2$ be the inclusion. If we let $M = K$ then we have that

$$0 \to \quad Hom_R(K, N_1) \quad \overset{(\phi_1)_*}{\longrightarrow} \quad Hom_R(K, N_2) \quad \overset{(\phi_2)_*}{\longrightarrow} \quad Hom_R(K, N_3)$$

is exact.

By construction, $(\phi_2) \circ i = 0$, i.e $(\phi_2)_*(i) = 0$, i.e. $i \in Ker(\phi_2)_*$. Thus, $i \in Im(\phi_1)_*$. But this means that there is an $f : Ker(\phi_2) \to N_1$ such that $(\phi_1)_*(f) = i$, i.e. $\phi_1 \circ f = i$. But since this composition is $1-1$ that means that f is $1-1$. Since ϕ_1 is also $1-1$ the composite shows that $Ker(\phi_2) \subseteq Im(\phi_1)$, as we wanted to show. □

Exercise 10 A: (10) Finish the proof of the proposition above.

End of Aside.

There is one other general proposition about tensor products which is often very useful.

Proposition: Let F be a free R-module with basis $\{\xi_i\}_{i \in I}$ and let N be an arbitrary R-module.

Then every element, x, of $F \otimes_R N$ can be *uniquely* written as a finite sum

$$x = \sum \xi_i \otimes n_i \ .$$

Proof: We know that $F \simeq \oplus_{i \in I} R\xi_i$ and, from our earlier comments, every $x \in F \otimes_R N$ can be written as a finite sum,

$$x = \sum \xi_i \otimes n_i \ .$$

So, to prove the proposition it suffices to show

$$\text{if } x_i = \xi_i \otimes n \quad \in \quad R\xi_i \otimes N$$

then n is uniquely determined by x_i.

We have an isomorphism $N \simeq R \otimes_R N$ given by $n \to 1 \otimes n$ and clearly $R \otimes_R N \simeq R\xi_i \otimes_R N$ given by $1 \otimes n \to \xi_i \otimes n$. The isomorphism $N \simeq R\xi_i \otimes N$ is thus given by $n \to \xi_i \otimes n$ and so we are done. $\square$.

Extension of the Base or Extension of Scalars

The tensor product can be used to describe a very ubiquitous situation in mathematics.

Let $f : R \to S$ be a homomorphism of commutative rings with identity and let M be an R-module.

Then we can think of S as an R-module via

$$rs := f(r)s \text{ where } r \in R \text{ and } s \in S \ .$$

The map

$$S \times S \times M \longrightarrow S \otimes_R M \text{ given by}$$

$$(x, y, m) \longrightarrow xy \otimes m$$

is easily seen to be an R-multilinear function. It thus defines a bilinear map

$$S \times (S \otimes_R M) \to S \otimes_R M$$

$$\text{by } (s, s' \otimes m) \longrightarrow ss' \otimes m \ .$$

But, that's the same thing as saying that $S \otimes_R M$ is an S-module under the action $s(s' \otimes m) = ss' \otimes m$.

So, we can view $S \otimes_R -$ as a functor from R-modules to S-modules. We'll denote this functor by $\mathcal{E}$ (for "extension" of scalars from R to S). (Of course, we need to check that it does the right thing to maps, i.e. that the new maps are S-linear, but I leave that easy verification to the reader.)

From what we have already seen about the $\otimes$, this is a right exact functor.

There is another natural functor in this circle of ideas, i.e. given $f : R \to S$ as above, and N an S-module, then N may be thought of as an R-module by

$$rn := f(r)n \text{ where } r \in R \text{ and } n \in N \ .$$

i.e. we are "forgetting" the S-action on N (except for those "s" in the image of f). It's not hard to show that this is a functor (a forgetful functor) which is exact!

Exercise 10B: (5) *a*) Verify that the functor is exact. Call it $\mathcal{R} : S - mod \to R - mod$, where I am thinking of "restricting" the scalars to R.

b) (10) Fix an R-module M. Prove that there is a natural isomorphism of functors between

$$Hom_R(M, \mathcal{R}(-)) \text{ and } \mathcal{R}(Hom_S(S \otimes_R M, -))$$

The functors going from S-modules to R-modules.

c) (10) Fix an S-module N. Prove that there is a natural isomorphism of functors between

$$\mathcal{E}(Hom_R(-, \mathcal{R}(N)) \text{ and } Hom_S(S \otimes_R -, N)$$

Exercise 10C: (10) Let $\phi : R \to S$ be a homomorphism of commutative rings with identity. If P is a projective R-module show that $P \otimes_R S$ is a projective S-module.

This possibility of considering a module as a module over more than one ring at a time is very useful. Let me be more precise and more general than what I have indicated above.

Let $\phi : R \to S$ be a homomorphism of commutative rings with identity and let M be an abelian group which, with its "additive" structure, is both an R and an S-module. (e.g. If N is any R-module then $N \otimes_R S$ is both an R and an S-module.)

Definition: We say that an M, as above, is an $(R - S) - bimodule$ if, in addition,

$$(rs)m = s(rm) = r(sm) \text{ for any } r \in R \text{ and } s \in S \ .$$

So, for example, $N \otimes_R S$ is an $(R - S)$-bimodule.

If M_1 and M_2 are both $(R - S)$-bimodules then $\phi : M_1 \to M_2$ is an $(R - S) -$ *homorphism* is it is both an R-module and S-module homomorphism.

Now, if A is an R-module, B an $(R - S)$-bimodule and C is an S-module then

Claim: i) $A \otimes_R B$ is an $(R-S)$-bimodule where

$$s(a \otimes b) = a \otimes sb \ ;$$

ii) $B \otimes_S C$ is an $(R-S)$-bimodule where

$$r(b \otimes c) = rb \otimes c \ .$$

These are both very easy verifications, which I will leave to you to check. The point of these comments is that, with the data above, we can form two distinct $(R-S)$-bimodules,

$$(A \otimes_R B) \otimes_S C \text{ and } A \otimes_R (B \otimes_S C) \ .$$

Our most useful general result in this setting is the following "associative" rule.

Theorem: With the data above, there is an $(R-S)$-bimodule isomorphism

$$(A \otimes_R B) \otimes_S C \simeq A \otimes_R (B \otimes_S C)$$

which takes $(a \otimes b) \otimes c \to a \otimes (b \otimes c)$.

Proof: Select $c \in C$ and keep it fixed. Consider

$$A \times B \longrightarrow A \otimes_R (B \otimes_S C) \text{ given by}$$

$$(a, b) \longrightarrow a \otimes (b \otimes c) \ .$$

Regarding everything as an R-module, it's easy to see this map is R-bilinear and so we have an R-module homomorphism which takes

$$a \otimes b \longrightarrow a \otimes (b \otimes c) \ .$$

Call this homomorphism, ϕ_c.

Now define

$$(A \otimes_R B) \times C \to A \otimes_R (B \otimes_S C)$$

by

$$(x, c) \longrightarrow \phi_c(x) \ .$$

This is easily seen to be an S-bilinear map. So, we get an S-module homomorphism

$$(a \otimes b) \otimes c \longrightarrow a \otimes (b \otimes c)$$

as we wanted. I.e. the map defined by $(a \otimes b) \otimes c \to a \otimes (b \otimes c)$ is an $(R-S)$-bimodule homomorphism.

As we can do the same thing, but working in the other direction (which I'll leave to the reader to check), we get that this map is an $(R-S)$-bimodule isomorphism. □

Sect.6B

With the theorem above proved, we get the following very useful consequence.

Proposition: Let $\phi : R \to S$ be a homomorphism of commutative rings with identity and let M and N be R-modules. There is an isomorphism of S-modules given by:

$$(M \otimes_R S) \otimes_S (N \otimes_R S) \simeq (M \otimes_R N) \otimes_R S$$

given by

$$(m \otimes s_1) \otimes (n \otimes s_2) \longrightarrow m \otimes n \otimes s_1 s_2 \ .$$

Proof: By our Theorem above, there is an $(R-S)$-isomorphism

$$M \otimes_R (S \otimes_S (N \otimes_R S)) \simeq (M \otimes_R S) \otimes_S (N \otimes_R S)$$

given by

$$m \otimes (s_1 \otimes (n \otimes s_2) \to (m \otimes s_1) \otimes (n \otimes s_2) \ .$$

By an earlier Proposition we have an isomorphism of S-modules

$$S \otimes_S (N \otimes_R S) \simeq N \otimes_R S$$

given by

$$s_1 \otimes (n \otimes s_2) \to s_1(n \otimes s_2) = n \otimes s_1 s_2 \ .$$

Notice that, in fact, in this case the isomorphism is an $(R-S)$-isomorphism.

Thus, combining these we get an $(R-S)$-isomorphism

$$(M \otimes_R S) \otimes_S (N \otimes_R S) \longrightarrow M \otimes_R (N \otimes_R S)$$

given by

$$(m \otimes s_1) \otimes (n \otimes s_2) \longrightarrow m \otimes (n \otimes s_1 s_2)\ .$$

We already know that we have an R-isomorphism

$$M \otimes_R (N \otimes_R S) \simeq (M \otimes_R N) \otimes_R S$$

given by

$$m \otimes (n \otimes s) \longrightarrow (m \otimes n) \otimes s$$

Once we note that that this last is also an S-module homomorphism, we get that it is an $(R-S)$-bimodule isomorphism. That is the final step needed to complete the proof. $\square$

There are a few easy remarks I would like to make about extension and restriction of scalars.

Proposition: Let $f : R \to S$ be a homomorphism of commutative rings, M an R-module and N an S-module.

i) If N is finitely generated as an S-module and S is finitely generated as an R-module then N is a finitely generated R-module.

ii) If M is a finitely generated R-module then $M \otimes_R S$ is a finitely generated S-module.

Proof: i) Let $n_1, \ldots, n_t$ generate N as an S-module and let $s_1, \ldots, s_v$ generate S as an R-module. Then

$$\{s_i n_j | 1 \leq i \leq v, q \leq j \leq t\} \text{ generate } N \text{ as an } R\text{-module.}$$

ii) If $m_1, \ldots, m_t$ generate M as an R-module then $\{1 \otimes m_1, \ldots, 1 \otimes m_t\}$ generate $S \otimes_R M$ as an S-module.

There are many interesting things about the tensor product that will come up as we study commutative rings in more detail. I want to indicate one of those theorems now. We'll see others later.

Theorem: Let (R, m) be a local ring with maximal ideal m and let M and N be finitely generated R-modules.

If $M \otimes_R N = 0$ then either $M = 0$ or $N = 0$.

Proof: In order to prove this theorem we shall need a fundamental result in the theory of local rings, which I will now prove:

Nakayama's Lemma: Let (R, m) be a local ring and let M be a finitely generated R-module. If I is an ideal of R and $IM = M$ then $M = 0$.

Proof: Let $M =< m_1, \ldots, m_t >$ be a set of generators for M. Since $IM = M$ we have,

$$\begin{array}{ccc} m_1 & = & a_{11}m_1 + \cdots + a_{it}m_t \\ \vdots & \vdots & \vdots \\ m_t & = & a_{t1}m_1 + \cdots + a_{tt}m_t \end{array}$$

where the $a_{ij} \in I$.

It follows that:

$$\begin{array}{rcc} (1 - a_{11})m_1 - \cdots - a_{it}m_t & = & 0 \\ \vdots & \vdots & \vdots \\ -a_{t1}m_1 - \cdots - (1 - a_{tt})m_t & = & 0 \end{array}$$

So, if we let

$$\mathcal{M} = \begin{pmatrix} 1 - a_{11} & -a_{12} & \cdots & -a_{1t} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{t1} & & \cdots & 1 - a_{tt} \end{pmatrix}$$

then

$$\mathcal{M} \begin{pmatrix} x_1 \\ \vdots \\ x_t \end{pmatrix} = \mathbf{0}$$

is a system of linear equations with a non-trivial solution in R.

But, $det(\mathcal{M}) = 1 + q$ where $q \in m$. I.e. the determinant is a unit of R. But that means that $\mathcal{M}$ is invertible over R and that is a contradiction to the fact that the system above has a non-trivial solution. $\square$

We now return to the proof of the theorem. The map $\eta : R \to R/m \simeq k$ is a homomorphism of commutative rings. So, since $M \otimes_R N = 0$ by assumption, we have $(M \otimes_R N) \otimes_R (R/m) = 0$.

But we just showed above, that

$$(M \otimes_R N) \otimes_R (R/m) \simeq (M \otimes_R (R/m)) \otimes_R (N \otimes_R (R/m)).$$

Since $M \otimes_R (R/m) \simeq M/mM$ and $N \otimes_R (R/m) \simeq N/mN$ are both vector spaces over the field $k = R/m$ and the tensor product of vector spaces is a vector space of dimension equal to the product of the dimensions of the factors, we must have one or the other of the factors $= 0$. But, by Nakayama's Lemma, that means that either M or N is 0. $\square$

There are three very important "ring homomorphisms" to which I would like to apply the results above.

$i)$ $R \longrightarrow S = R/I$, where I is an ideal of R;

$ii)$ $R \longrightarrow R[x]$, the inclusion of R into the polynomial ring in one indeterminate over R;

$iii)$ $R \longrightarrow S^{-1}R$, where S is a multiplicatively closed subset of R which contains the 1 of R.

As for the first example, we have seen this considered earlier, since the passage

$$M \longrightarrow R/I \otimes_R M \simeq M/IM$$

So, we are really just looking at the assignment

$$M \longrightarrow M/IM$$

and what we proved above shows that this assignment is a right exact functor.

Remark: One might reasonably wonder when R/I is a flat R-module, so that the assignment $M \longrightarrow M/IM$ is an exact functor. I don't want to go into much detail here except to say that it is a very rare occurence! I'll record here one small result which shows how hard it is for R/I to be R-flat.

Claim: Let I be an ideal of R and suppose that I contains a non-zero divisor a which is not in I^2. Then R/I is not R-flat.

Proof: With a as above, we have an exact sequence

$$0 \longrightarrow \quad I \quad \stackrel{\times a}{\longrightarrow} \quad I \rightarrow \quad I/aI \quad \longrightarrow 0$$

It will be enough to show that

$$I \otimes_R (R/I) \quad \stackrel{a \otimes id}{\longrightarrow} \quad I \otimes_R (R/I)$$

is not $1-1$.

But this map is just

$$I/I^2 \quad \stackrel{\times a}{\longrightarrow} \quad I/I^2$$

and since a is not in I^2, the map takes the class $\overline{a}$ to zero. So, the map is not $1-1$. □

As for the second example above, notice that $R[x]$ is free as an R-module and hence is a flat R-module. Thus, the functor $- \otimes_R R[x]$ is an exact functor.

What does this functor do? Let M be an R-module, then $M \otimes_R R[x]$ can be thought of as follows:

since $R[x] \simeq R1 \oplus Rx \oplus Rx^2 \oplus \cdots$ then tensoring this (over R) with M will give $M1 \oplus Mx \oplus Mx^2 \oplus \cdots$, i.e. we can think of $M \otimes_R R[x]$ as polynomials with coefficients in M.

In fact we often write $M[x]$ instead of $M \otimes_R R[x]$.

The last of the homomorphisms is, in some ways, the most interesting.

Recall that if S is a multiplicatively closed subset of R which contains 1 then we have seen how to construct $S^{-1}R$. We can mimic this for any R-module M as follows.

Consider

$$\{m/s \mid m \in M \text{ and } s \in S\}$$

We put an equivalence relation on these "fractions" by saying

$$m_1/s_1 = m_2/s_2 \Leftrightarrow \text{ there is an } s_3 \in S \text{ such that } s_3(s_2 m_1 - s_1 m_2) = 0 \ .$$

It is easy to show that this is an equivalence relation on the set above and if we use $[m/s]$ to describe the equivalence class containing m/s then the set of equivalence classes so obtained is denoted $S^{-1}M$.

As with the case of $M = R$ it is easy to show that

$$[m_1/s_1] + [m_2/s_2] = [(s_2 m_1 + s_1 m_2)/(s_1 s_2)]$$

gives a well-defined addition on $S^{-1}M$ and that

$$[r_1/s_1][m_1/s_2] = [r_1 m_1/s_1 s_2]$$

is also a well-defined multiplication by elements of $S^{-1}R$ which makes $S^{-1}M$ into an $S^{-1}R$-module.

Proposition: With the notation above,

$$S^{-1}M \simeq S^{-1}R \otimes_R M \ .$$

Proof: Consider the function

$$S^{-1}R \times M \longrightarrow S^{-1}M \text{ given by } (r/s, m) \longrightarrow (rm)/s \ .$$

This is easily seen to be R-bilinear and so gives an R-linear map

$$\phi : S^{-1}R \otimes_R M \longrightarrow S^{-1}M$$

which is clearly onto.

Let $x \in S^{-1}R \otimes_R M$, $x = \sum_{i=1}^n ((r_i/s_i) \otimes m_i)$. Let $s = \Pi_{i=1}^n s_i$ and let $t_i = \Pi_{j \neq i} s_j$. Then

$$x = \sum_{i=1}^n ((r_i t_i)/s) \otimes m_i = \sum_{i=1}^n (1/s \otimes r_i t_i m_i) = (1/s) \otimes \sum_{i=1}^n (r_i t_i m_i) = (1/s) \otimes m$$

for some $m \in M$.

So, suppose that $\phi(1/s \otimes m) = 0$, i.e. $m/s = 0$. Then there is some $t \in S$ such that $tm = 0$. Thus, $(1/s) \otimes m = (t/ts) \otimes m = (1/ts) \otimes tm = 0$, as we wanted to show. $\square$

To continue the symmetry of our discussion we consider when $S^{-1}R$ is a flat R-module. The answer is : always!

Proposition: Let S and R be as above. Then $S^{-1}R$ is a flat R-module.

Proof: Since the functor $- \otimes_R S^{-1}R$ is the same as taking S^{-1} of a module.

So, it's enough to show that if

$$0 \longrightarrow M_1 \xrightarrow{f} M_2$$

is exact, then

$$0 \longrightarrow S^{-1}M_1 \xrightarrow{f} S^{-1}M_2$$

is exact. I.e. we want to know if the assignment $m/s \to f(m)/s$ is $1-1$.

Let $m/s \in S^{-1}M_1$ be such that $f(m)/s = 0 \in S^{-1}M_2$. Then there is an element $t \in S$ such that $tf(m) = 0$. But then $f(tm) = 0$. Since f is $1-1$ then $tm = 0$. But then $m/s = (tm)/(ts) = 0/s = 0$, as we wanted to show. $\square$

sect7a.tex

Chain Complexes of R-modules

Let $\mathcal{C}h_R$ denote the category of chain complexes of R-modules (a "co-discussion" of all of this for cochain complexes can be made, which I won't give, but which I expect you to work out). Recall that, for each spot in the chain, we defined an homology module as follows: if $(A_\bullet, d_\bullet)$ is an object of $\mathcal{C}h_R$, i.e. we have

$$\cdots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \to \cdots$$

then $H_n(A_\bullet) = Ker(d_n)/Im(d_{n+1})$.

The submodule, $Ker(d_n)$ of A_n, is denoted $Z_n(A_\bullet)$ and its elements are called *n-cycles* of $A_\bullet$. The submodule, $Im(d_{n+1})$ of A_n is denoted $B_n(A_\bullet)$ and its elements are called *n-boundaries* of $A_\bullet$.

Proposition: The assignment $(A_\bullet, d_\bullet) \to H_n(A_\bullet)$ is a functor from $\mathcal{C}h_R$ to R-mod.

Proof: We know what this functor does to objects but we've not said what it does to morphisms.

So, let $f_\bullet : (A_\bullet, d_\bullet) \to (M_\bullet, d'_\bullet)$, then, in particular, we have

$$\begin{array}{ccc} A_n & \xrightarrow{f_n} & M_n \\ d_n \downarrow & & \downarrow d'_n \\ A_{n-1} & \xrightarrow{f_{n-1}} & M_{n-1} \end{array}$$

If $x \in Ker(d_n)$ then $f_{n-1}d_n(x) = f_{n-1}(0) = 0$. Thus, $d'_n f_n(x) = 0$ as well. Hence, f_n induces a map, also denoted f_n, given by $f_n : Z_n(A_\bullet) \longrightarrow Z_n(M_\bullet)$.

Claim: $f_n(B_n(A_\bullet)) \subseteq B_n(M_\bullet)$.
Pf: Let $x \in B_n(A_\bullet)$, i.e. $x = d_{n+1}(y)$, where $y \in A_{n+1}$. So, we have (graphically)

$$\begin{array}{ccc} y & \to & f_{n+1}(y) \\ \downarrow & & \downarrow \\ x & \to & f_n(x) \end{array}$$

i.e. $f_n(x) = d'_{n+1}(f_{n+1}(y))$, i.e. $f_n(x) \in B_n(M_\bullet)$. $\square$

Thus, f_n induces a map $H_n(A_\bullet) \to H_n(M_\bullet)$ which I'll denote by $(f_n)_*$ or $H_n(f_\bullet)$, given by

$$H_n(f_\bullet) = (f_n)_* : \overline{x} \to \overline{f_n(x)} \text{ where } x \in Z_n(A_\bullet)\ .$$

Now, to show that $H_n : \mathcal{C}h \to$R$-mod$ is a functor, all we really need to see is that

$$H_n(id_\bullet) = id \text{ and}$$
$$H_n(f_\bullet g_\bullet) = H_n(f_\bullet)H_n(g_\bullet).$$

But, given the description of $H_n(f_\bullet)$ above, this is obvious. $\square$

Now, it turns out that the functors H_n have a "small" exactness property, which is very useful.

Proposition: If

$$0 \to (A_\bullet, d_\bullet) \xrightarrow{f_\bullet} (M_\bullet, d'_\bullet) \xrightarrow{g_\bullet} (L_\bullet, d''_\bullet) \to 0$$

is a short exact sequence of chain complexes, then

$$H_n(A_\bullet) \xrightarrow{H_n(f_\bullet)} H_n(M_\bullet) \xrightarrow{H_n(g_\bullet)} H_n(L_\bullet)$$

is exact. (This is sometimes called "middle" exact.)

Proof: It's enough to prove

$$Ker(H_n(g_\bullet)) \subseteq Im(H_n(f_\bullet)) \ .$$

So, let $\overline{m_n} \in H_n(M_\bullet)$ and suppose that $(g_n)_*(\overline{m_n}) = 0$, i.e. $\overline{g_n(m_n)} = 0$ in $H_n(L_\bullet)$. Thus, $g_n(m_n) \in B_n(L_\bullet)$. By definition, $g_n(m_n) = d''_{n+1}(\ell_{n+1})$ and, since g_{n+1} is onto, $\ell_{n+1} = g_{n+1}(m_{n+1})$, so we have

$$\begin{array}{ccc} m_{n+1} & \to & \ell_{n+1} \\ & & \downarrow \\ m_n & \to & g_n(m_n) \end{array}$$

Consider $m_n - d'_{n+1}(m_{n+1}) \in M_n$. We have

$$g_n(m_n - d'_{n+1}(m_{n+1})) = g_n(m_n) - g_n(m_n) = 0 \ .$$

Since $Ker(g_\bullet) = Im(f_\bullet)$ this implies that $m_n - d'_{n+1}(m_{n+1}) = f(a_n)$ for some $a_n \in A_n$. We'll be done as soon as we show that $a_n \in Z_n(A_\bullet)$.

But, observe that we have

$$\begin{array}{ccc} a_n & \to & m_n - d'_{n+1}(m_{n+1}) \in Z_n(M_\bullet) \\ \downarrow & & \downarrow \\ x & \xrightarrow{f_{n+1}} & 0 \end{array}$$

Since f_{n+1} is $1-1$ that implies that $x = 0$, i.e. $a_n \in Z_n(A_\bullet)$, as we wanted. $\square$

So, given a short exact sequence of chain complexes

$$0 \to A_\bullet \xrightarrow{f_\bullet} M_\bullet \xrightarrow{g_\bullet} L_\bullet \to 0$$

we have a collection of exact sequences

$$\{ \ H_n(A_\bullet) \overset{H_n(f_\bullet)}{\longrightarrow} H_n(M_\bullet) \overset{H_n(g_\bullet)}{\longrightarrow} H_n(L_\bullet) \ \} \ , \tag{$*$}$$

one for each integer n. There seems nothing to connect these – or is there?

From the short exact sequence of chain complexes $(*)$ above, we have a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Z_n(A_\bullet) & \overset{f_n}{\longrightarrow} & Z_n(M_\bullet) & \overset{g_n}{\longrightarrow} & Z_n(L_\bullet) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_n & \overset{f_n}{\longrightarrow} & M_n & \overset{g_n}{\longrightarrow} & L_n & \to & 0 \\
 & & d_n \downarrow & & \downarrow d'_n & & \downarrow d''_n & & \\
0 & \to & A_{n-1} & \overset{f_{n-1}}{\longrightarrow} & M_{n-1} & \overset{g_{n-1}}{\longrightarrow} & L_{n-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A_{n-1}/d_nA_n & \overset{\widetilde{f_{n-1}}}{\longrightarrow} & M_{n-1}/d'_nM_n & \overset{\widetilde{g_{n-1}}}{\longrightarrow} & L_{n-1}/d''_nL_n & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{$**$}$$

Recall that there an induced map,

$$\partial_n : Z_n(L_\bullet) \longrightarrow A_{n-1}/d_nA_n \ ,$$

defined as follows: let $\ell_n \in Z_n(L_\bullet)$, then $\partial_n(\ell_n)$ was defined as follows:

$$\begin{array}{ccccc}
 & & m_n & \longrightarrow & \ell_n \\
 & & \downarrow & & \\
a_{n-1} & \longrightarrow & m_{n-1} & & \\
\downarrow & & & & \\
\overline{a_{n-1}} & & & &
\end{array}$$

where $m_n \in M_n$ was an element such that $g_n(m_n) = \ell_n$. I.e. $\partial_n(\ell_n) = \overline{a_{n-1}}$.

Claim: Any element a_{n-1}, as defined above, is in $Z_{n-1}(A_\bullet)$.

Pf: It's enough to show that $d_{n-1}(a_{n-1}) = 0$. But, in the diagram above we have

$$\begin{array}{ccccc} & & m_n & \longrightarrow & \ell_n \\ & & \downarrow & & \\ a_{n-1} & \longrightarrow & m_{n-1} & & \\ \downarrow & & \downarrow & & \\ x & \cdots\to & 0 & & \end{array}$$

i.e. $d_{n-1}(a_{n-1}) = x$ and $f_{n-2}(x) = 0$. Since f_{n-2} is $1-1$ this implies that $x = 0$. $\square$

Thus, $\partial_n : Z_n(L_\bullet) \to H_{n-1}(A_\bullet)$.

Claim: $\partial_n(B_n(L_\bullet)) = 0$.

Pf: Let $\ell_n \in B_n(L_\bullet)$, i.e. $\ell_n = d''_{n+1}(\ell_{n+1})$. Then we have the following picture:

$$\begin{array}{ccccc} & & m_{n+1} & \longrightarrow & \ell_{n+1} \\ & & \downarrow & & \downarrow \\ & & m_n & \longrightarrow & \ell_n \\ & & \downarrow & & \\ 0_{n-1} & \longrightarrow & 0_{n-1} & & \\ \downarrow & & & & \\ \overline{0_{n-1}} & & & & \end{array}$$

and we are done. $\square$

So, we have a collection of "*connecting*" R-module homomorphisms,

$$H_{n+1}(L_\bullet) \xrightarrow{\partial_{n+1}} H_n(A_\bullet)$$

one for each n.

Theorem: Let

$$0 \longrightarrow A_\bullet \xrightarrow{f_\bullet} M_\bullet \xrightarrow{g_\bullet} L_\bullet \longrightarrow 0$$

be a short exact sequence of chain complexes of R-modules.

Then there is a long exact sequence of R-modules

$$\cdots \stackrel{H_{n+1}(g_\bullet)}{\to} H_{n+1}(L_\bullet) \stackrel{\partial_{n+1}}{\to} H_n(A_\bullet) \stackrel{(f_n)_*}{\to} H_n(M_\bullet) \stackrel{(g_n)_*}{\to} H_n(L_\bullet) \stackrel{\partial_n}{\to} H_{n-1}(A_\bullet) \cdots$$

for each n.

Proof: From the Snake Lemma and the diagram $(**)$ above we have a complex since

$$H_{n-1}(A_\bullet)/B_{n-1}(A_\bullet) \subseteq A_{n-1}/d_n A_n$$

and the map $\widetilde{f_{n-1}}$ restricted to $H_{n-1}(A_\bullet)$ is what we called $H_{n-1}(f_\bullet)$.

Thus to prove the theorem it is enough to show

$i)$ $Ker(\partial_n) \subseteq Im(H_n(g_\bullet))$ and

$ii)$ $Ker(H_{n-1}(f_\bullet)) \subseteq Im(\partial_n)$ for each n.

As for $i)$, suppose that $\partial_n(\overline{\ell_n}) = 0$, where $\ell_n \in Z_n(L_\bullet)$. So, we have

$$\begin{array}{ccccc} & & m_n & \to & \ell_n \\ & & \downarrow & & \\ a_{n-1} & \to & m_{n-1} & & \\ \downarrow & & & & \\ \overline{a_{n-1}} & & & & \end{array}$$

where $m_n \in M_n$ and $a_{n-1} \in d_n A_n$ (since that is the meaning of $\partial_n(\overline{\ell_n}) = 0$). So, we also have the diagram

$$\begin{array}{ccc} a_n & \to & m'_n \\ \downarrow & & \downarrow \\ a_{n-1} & \to & m_{n-1} \end{array}$$

Consider $m_n - m'_n$. Then $d_n(m_n - m'_n) = 0$ since m'_n and m_n have the same image m_{n-1}. Thus, $m_n - m'_n \in Z_n(M_\bullet)$.

Now, $g_n(m_n - m'_n) = g_n(m_n) - g_n(m'_n) = \ell_n - 0$ since $m'_n = f_n(a_n)$ and $g_n f_n(a_n) = 0$. Thus, $\ell_n \in Im((g_n)_*)$ and that finishes this discussion.

As for $ii)$, let $\overline{x} \in Ker(f_{n-1})_*$, i.e. $\overline{f_{n-1}(x)} = \overline{0}$ in $H_{n-1}(M_\bullet)$, i.e. $x \in Z_{n-1}(A_\bullet)$ and $f_{n-1}(x) \in B_{n-1}(M_\bullet)$. Thus, $f_{n-1}(x) = d'_n(m_n)$. So, if we let $\ell_n = g_n(m_n)$ we have the following situation:

$$\begin{array}{ccc} & m_n & \rightarrow \quad \ell_n \\ & \downarrow & \\ x \quad \rightarrow & f_{n-1}(x) & \end{array}$$

Now, $\partial_n(\ell_n) = \overline{x}$, by definition, so the only problem left is that we are not sure if $\ell_n \in Z_n(L_\bullet)$.

To see that this is so, we consider the last diagram, augmented:

$$\begin{array}{ccc} & m_n & \rightarrow \quad \ell_n \\ & \downarrow & \\ x \quad \rightarrow & f_{n-1}(x) & \xrightarrow{g_n} \quad 0 \end{array}$$

But then, by the commutativity of the diagrams above, $d''_n(\ell_n) = 0$, so $\ell_n \in Z_n(L_\bullet)$ as we wanted to show.

That completes the proof of the theorem. $\square$

Exercise 13A: $a)$ (5) Let

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

be a short exact sequence of chain complexes. Prove that if any two of these chain complexes is exact then so is the third.

$b)$ (5) Show that the functors $H_n(-) : \mathcal{C}h \to R$-mod are all additive functors.

The naturality of the connecting morphism

I'd like to explain in what sense the connecting morphisms can be described as "natural".

Consider the category whose objects are : short exact sequences of chain complexes of R-modules and whose morphisms are triples of morphisms of chain complexes, i.e.

$$\begin{array}{ccccccccc} 0 & \to & A_\bullet & \xrightarrow{f_\bullet} & M_\bullet & \xrightarrow{g_\bullet} & L_\bullet & \to & 0 \\ & & \alpha_\bullet \downarrow & & \beta_\bullet \downarrow & & \gamma_\bullet \downarrow & & \\ 0 & \to & A'_\bullet & \xrightarrow{f'_\bullet} & M'_\bullet & \xrightarrow{g'_\bullet} & L'_\bullet & \to & 0 \end{array} \qquad (\dagger)$$

Moreover, for each n, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A_n & \xrightarrow{f_n} & M_n & \xrightarrow{g_n} & L_n & \to & 0 \\ & & \alpha_n \downarrow & & \beta_n \downarrow & & \gamma_n \downarrow & & \\ 0 & \to & A'_n & \xrightarrow{f'_n} & M'_n & \xrightarrow{g'_n} & L'_n & \to & 0 \end{array}$$

Now, there is yet another category, the category of Long Exact Sequence of R-modules. As you might suspect, the objects of this category are long exact sequences of R-modules and the morphisms are families of R-module homomorphisms between these sequences so that all the appropriate squares commute.

With those observations made, we can now state the meaning of the "naturality".

Theorem: Let $\mathcal{A}$ be the category of short exact sequences of chain complexes of R-modules and let $\mathcal{B}$ be the category of long exact sequences of R-modules.

Then, the long exact sequence of homology is a functor

$$\mathcal{H} : \mathcal{A} \longrightarrow \mathcal{B} \ .$$

Proof: I.e. given a diagram like $(\dagger)$ above, we have the following:

$$\begin{array}{cccccccccc} \cdots & \stackrel{\partial}{\to} & H_n(A_\bullet) & \stackrel{H_n(f_\bullet)}{\to} & H_n(M_\bullet) & \stackrel{(g_n)_*}{\to} & H_n(L_\bullet) & \stackrel{\partial_n}{\to} & H_{n-1}(A_\bullet) & \cdots \\ & & (\alpha_n)_* \downarrow & & (\beta_n)_* \downarrow & & (\gamma_n)_* \downarrow & & (\alpha_{n-1})_* & \\ \cdots & \stackrel{\partial}{\to} & H_n(A'_\bullet) & \stackrel{H_n(f_\bullet)}{\to} & H_n(M'_\bullet) & \stackrel{(g_n)_*}{\to} & H_n(L'_\bullet) & \stackrel{\partial_n}{\to} & H_{n-1}(A'_\bullet) & \cdots \end{array}$$

and we must just show that the squares commute.

Now, the first two squares commute because the H_n are functors. The only problem is the *rightmost square*. I'll leave that as a simple exercise.

Sect7b.tex

Chain Homotopies

We've seen that if $A_\bullet$ and $M_\bullet$ are in $\mathcal{C}h_R$ and $f_\bullet : A_\bullet \to M_\bullet$ then each of the functors H_n induces a map

$$H_n(f_\bullet) : H_n(A_\bullet) \to H_n(M_\bullet) \ .$$

Clearly, if $f_\bullet$ is an isomorphism of chain complexes, then the maps $H_n(f_\bullet)$ must be isomorphisms of R-modules for each n.

But, as we'll see, one doesn't need quite so much to get that the maps $H_n(f_\bullet)$ are all isomorphisms. First a simple

Remark: Let $A_\bullet, M_\bullet$ be two chain complexes and let $s_n : A_n \to M_{n+1}$ be *any* R-module homomorphism. If we use d_n to denote the chain maps in $A_\bullet$ and d'_n to denote the chain maps in $M_\bullet$, then

Claim: The maps

$$f_n = d'_{n+1} \circ s_n + s_{n-1} \circ d_n$$

are chain maps from $A_\bullet$ to $M_\bullet$.

Pf. It's enough to show that

$$f_{n-1} \circ d_n = d'_n \circ f_n \ .$$

But,

$$f_{n-1} \circ d_n = (d'_n \circ s_{n-1} + s_{n-2} \circ d_{n-1}) \circ d_n = d'_n \circ s_{n-1} \circ d_n$$

and this clearly equals $d'_n \circ (d'_{n+1} s_n + s_{n-1} d_n)$ as we wanted to show.

$\square$

We are now ready for the main Lemma.

Lemma: Let the chain map $f_\bullet = \{f_n\}$ be as defined above. Then $H_n(f_\bullet) = 0$.

Proof: Let $\overline{x} \in H_n(A_\bullet)$, i.e. $x \in Z_n(A_\bullet)$. So, we have $d_n(x) = 0$.

Now,

$$H_n(f_\bullet)(\overline{x}) = \overline{f_n(x)} = \overline{(d'_{n+1} \circ s_n + s_{n-1} \circ d_n)(x)}$$

Since $d_n(x) = 0$, we have

$$H_n(f_\bullet)(\overline{x}) = \overline{(d'_{n+1} \circ s_n)(x)} = \overline{d'_{n+1}(s_n(x))}$$

Since this last expression is in $B_n(M_\bullet)$, we are done. $\square$

Definition: Let $f_\bullet$, $g_\bullet$ be two chain maps from $A_\bullet$ to $M_\bullet$. We say that $f_\bullet$ and $g_\bullet$ are *chain homotopic* if there are R-module homomorphisms $\{s_n\}$, where $s_n : A_n \to M_{n+1}$ such that

$$(f_\bullet - g_\bullet)_n = d'_{n+1} s_n + s_{n-1} d_n \ .$$

Remark: If $f_\bullet$ and $g_\bullet$ are chain homotopic then (since H_n is an additive functor) the Lemma above implies that $H_n(f_\bullet) = H_n(g_\bullet)$. So, anything which is chain homotopic to an isomorphism of chain complexes induces isomorphisms on homology.

So, e.g. if we let $s_n : A_n \to M_{n+1}$ be *any* family of R-module homomorphisms and define the maps

$$\alpha_n = d'_{n+1} s_n + s_{n-1} d_n$$

then these maps can be added to, for example, the identity map of chain complexes, to give a map of chain complexes which induces the identity map on homology.

We shall have occasion to use this latter.

Exercise 14 A: (10) Chain homotopy is an equivalence relation on the chain maps from $A_\bullet \to B_\bullet$ in $\mathcal{C}h_R$.

Derived Functors

We now come to the heart of a course on homological algebra, the place where a connection is made between the functors we have been looking at so far and the notions of homology coming from algebraic topology. This discussion has undergone many revisions since it was introduced in the 1940's by Cartan and Eilenberg, but the basic ideas remain, with only the exposition being cleaned up over time.

Definition: A *covariant homological δ-functor* (respectively *covariant cohomological δ-functor*) between the abelian categories $\mathfrak{A}$ and $\mathfrak{B}$, is a collection of additive functors

$$T_n : \mathfrak{A} \to \mathfrak{B} \text{ (respectively } T^n : \mathfrak{A} \to \mathfrak{B} \text{)}$$

(where, for $n < 0$ the convention is that T_n (respectively T^n) is the 0-functor)

such that, for every short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

in $\mathfrak{A}$ there is a morphism

$$\delta_n : T_n(C) \to T_{n-1}(A) \text{ (respectively } \delta^n : T^n(C) \to T^{n+1}(A))$$

such that

i) The following long sequence is exact in $\mathfrak{B}$,

$$\cdots \to T_{n+1}(C) \xrightarrow{\delta_{n+1}} T_n(A) \xrightarrow{T_n(f)} T_n(B) \xrightarrow{T_n(g)} T_n(C) \xrightarrow{\delta_n} T_{n-1}(A) \to \cdots$$

(respectively,

$$\cdots \to T^{n-1}(C) \xrightarrow{\delta^{n-1}} T^n(A) \xrightarrow{T^n(f)} T^n(B) \xrightarrow{T^n(g)} T^n(C) \xrightarrow{\delta^n} T^{n+1}(A) \to \cdots \text{).}$$

ii) For every morphism between short exact sequences in $\mathfrak{A}$,

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & , \\ 0 & \to & A' & \to & B' & \to & C' & \to & 0 \end{array}$$

there is a commutative diagram

$$\begin{array}{ccc} T_{n+1}(C) & \xrightarrow{\delta_n} & T_n(A) \\ T_{n+1}(\gamma)\downarrow & & \downarrow T_n(\alpha) \\ T_{n+1}(C') & \xrightarrow{\delta_n} & T_n(A') \end{array}$$

(respectively,

$$\begin{array}{ccc} T^n(C) & \xrightarrow{\delta^n} & T^{n+1}(A) \\ T^n(\gamma)\downarrow & & \downarrow T^{n+1}(\alpha) \\ T^n(C') & \xrightarrow{\delta^n} & T^{n+1}(A') \end{array} \quad) .$$

Examples:

1). If $\mathfrak{A} = \mathcal{C}h_R^{\geq 0}$ (i.e. chain complexes $C_\bullet$ for which $C_n = 0$ for $n < 0$) and $\mathfrak{B} = R$-mod, then the homology functors $H_n : \mathfrak{A} \to \mathfrak{B}$ form a covariant homological δ-functor.

2). Let $\mathfrak{A}$ be Abelian groups and let $\mathfrak{B}$ be the same category. If p is any integer, then pG is a well-defined subgroup of the group G.

Let T_0 be the functor which takes the group G to the group G/pG. If f is a group homomorphism from A to B then it is easy to see that $f(pA) \subseteq pB$, so there is always an induced map from A/pA to B/pB induced by f. So, this is a functor. It is easy to see that T_0 is an additive functor.

Let T_1 be the functor which takes the group G to the subgroup ${}_pG$, where

$${}_pG = \{g \in G \mid pg = 0\} .$$

This is also a functor which is easily seen to be additive.

Claim: $\{T_0\ , T_1\}$ form a homological δ-functor from **Ab** to **Ab**.

Pf: Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be an exact sequence of abelian groups. Then the diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 \\ & & \times p\downarrow & & \times p\downarrow & & \times p\downarrow & & \\ 0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 \end{array}$$

is a commuting diagram.

Thus, from the Snake Lemma we get

$$0 \to_p A \to_p B \to_p C \overset{\delta}{\to} A/pA \to B/pB \to C/pC \to 0$$

so, the connecting morphism does the trick.

It remains to check the "naturality" of the connecting morphism, and I shall leave that as an exercise.

Definition: 1) A morphism $\{S_n\} \to \{T_n\}$ of covariant homological δ-functors from $\mathfrak{A}$ to $\mathfrak{B}$ is a system of natural transformations $S_n \to T_n$ that commute with the δ's. I.e. if η_n is the natural transformation from S_n to T_n and η_{n-1} is the natural transformation from S_{n-1} to T_{n-1} and $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathfrak{A}$, then we have a commuting diagram

$$\begin{array}{ccc} S_{n-1}(A) & \overset{\eta_{n-1}(A)}{\longrightarrow} & T_{n-1}(A) \\ \delta\uparrow & & \delta\uparrow \\ S_n(C) & \overset{\eta_n(C)}{\longrightarrow} & T_n(C) \end{array}$$

What this amounts to is the following: given $0 \to A \to B \to C \to 0$ a short exact sequence in $\mathfrak{A}$, then we have the following commutative diagram:

$$\begin{array}{ccccccccccc} \cdots & S_n(A) & \to & S_n(B) & \to & S_n(C) & \overset{\delta}{\to} & S_{n-1}(A) & \to & S_{n-1}(B) \\ & \eta_n(A)\downarrow & & \eta_n(B)\downarrow & & \eta_n(C)\downarrow & & \eta_{n-1}(A)\downarrow & & \eta_{n-1}(B)\downarrow \\ \cdots & T_n(A) & \to & T_n(B) & \to & T_n(C) & \overset{\delta}{\to} & T_{n-1}(A) & \to & T_{n-1}(B) \end{array}$$

Notice that most of these squares commute because of the definition of natural transformation of functors. The new thing is that the square with the δ in it also commutes.

2) A covariant homological δ-functor, $T = \{T_n\}$, is *universal* if, given any other covariant homological δ-functor, $S = \{S_n\}$ and a natural transformation $\eta_0 : S_0 \to T_0$, there is a **unique** morphism of δ-functors which extends η_0.

Just to convince you that such esoteric objects actually exist, consider the following:

Proposition: Let $F : \mathfrak{A} \to \mathfrak{B}$ be a covariant **exact** functor.

Then $T = \{T_0 = F, T_n = 0 \text{ for } n > 0\}$ is a universal homological δ-functor.

Proof: If $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathfrak{A}$, then, since F is exact, we get the long exact sequence:

$$\cdots \to 0 \to 0 \to F(A) \to F(B) \to F(C) \to 0$$

So, $\delta_n = 0$ works for all $n \geq 0$. The universality is obvious.

Sect8a.tex

The question arises: if $F : \mathfrak{A} \to \mathfrak{B}$ is an additive functor, is there **ANY** δ-functor T (universal or not!) from $\mathfrak{A}$ to $\mathfrak{B}$ for which $T_0 = F$?

An obvious restriction (since $T_n = 0$ for $n < 0$) is that T_0 be right exact. (Similarly, for a covariant cohomological δ-functor, T^0 is left exact).

If, given F, a universal such T exists, then the T_n are called the "left satellites" of T_0 (respectively, the "right satellites" of T^0,)

We will show that, under certain circumstances, "left satellites" exist. I shall only do this for a functor from R-modules to R-modules. It is possible to do this much more generally. It will be clear that a special role is played by the projective R-modules. Since projective R-modules are defined "categorically" (i.e. in terms of diagrams and epis and zero objects), it is possible to define "projective objects" in any abelian category. The KEY thing we will continue to use is the fact that for any R-module there is a projective R-module mapping onto it. The analogous idea in an abelian category will be the idea of the category having "enough projectives". I suggest you read this in Weibel, or any other book on category theory. I won't go into it in much detail now. There is an analogous notion of an abelian category having "enough injectives".

We've spoken about covariant homological and cohomological δ-functors and we have seen that each of them is "based" on, in the first instance, a right exact functor and, in the second instance, a left exact functor.

Our two principal examples will be the right exact functors:

$$- \otimes_R M : R - mod \longrightarrow R - mod$$

where M is an R-module, and the left exact functors

$$Hom_R(M,-):\ R-mod \to R-mod\ .$$

So, we'll start as follows: let F be a covariant right exact functor (a potential F_0, as part of a covariant, homological δ-functor) where $F: R-\text{mod} \to R-\text{mod}$.

Definition: Let M be any R-module. A *left resolution* of M is a (chain) complex $(P_\bullet, d_\bullet)$ where $P_i = 0$ for $i < 0$, along with an R-module homomorphism $\epsilon: P_0 \to M$ (called the augmentation map) so that the complex

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0$$

is exact.

If each P_i is a projective R-module then $(P_\bullet, d_\bullet, \epsilon)$ is called a *projective* resolution of M.

Lemma: Every R-module M has a projective resolution.

Proof: First, we can always find a projective R-module P_0 and a homomorphism ϵ_0 such that $\epsilon_0: P_0 \twoheadrightarrow M$. Let $M_0 = Ker(\epsilon_0)$. We have

$$\begin{array}{ccccccccc}
P_1 & & & & P_0 & \overset{\epsilon_0}{\to} & M & \to & 0 \\
& \searrow \epsilon_1 & & i_0 \nearrow & & & & & \\
& & Ker(\epsilon_0) & & & & & & \\
& \nearrow & & \searrow & & & & & \\
0 & & & & 0 & & & &
\end{array}$$

where P_1 is a projective R-module and ϵ_1 a surjective homomorphism onto $Ker(\epsilon_0)$. Then (abusively) $\epsilon_1 = i_0\epsilon_1 : P_1 \to P_0$ and $Im(i_0\epsilon_0) = Ker(\epsilon_0)$.

So,

$$P_1 \xrightarrow{\epsilon_1} P_0 \xrightarrow{\epsilon_0} M \to 0$$

is exact. We continue this process. $\square$

Now that we know projective resolutions exist we think about mappings between them.

Comparison Theorem: Let $(P_\bullet, d_\bullet \epsilon)$ be a projective resolution of the R-module M and let $f : M \to N$ be a homomorphism of R-modules.

If $(Q_\bullet, d'_\bullet, \eta)$ is any resolution of N, then there is a chain map $f_\bullet$,

$$f_\bullet : (P_\bullet, d_\bullet) \to (Q_\bullet, d'_\bullet)$$

which "lifts" f, i.e. for which we also have $f\epsilon = \eta f_0$

Moreover, any two chain maps as above, which lift f, are chain homotopic.

Remarks:
1) It's not apparent, yet, how this result about chain homotopy is interesting since $(P_\bullet, d_\bullet)$ and $(Q_\bullet, d'_\bullet)$ are exact everywhere, except possibly at P_0 and Q_0.
2) As we'll see from the proof, the only hypothesis we really need on $(P_\bullet, d_\bullet)$ is that it is a complex of projective R-modules. Keep that in mind for later.

Proof: Let's get the construction started. We will then proceed by induction. We have a homomorphism f_0 where

$$\begin{array}{ccccc} P_0 & \overset{\epsilon}{\twoheadrightarrow} & M & \to & 0 \\ f_0 \downarrow & & \downarrow f & & \\ Q_0 & \overset{\eta}{\twoheadrightarrow} & N & \to & 0 \end{array}$$

since P_0 is a projective R-module and η is surjective. Moreover, $f\epsilon = \eta f_0$. So, the process of constructing a chain map may be begun.

So, by induction, suppose we have constructed f_i, for $i \leq n$, which satisfies the properties of a chain map. So, we have:

$$\begin{array}{ccc} P_{n+1} & \overset{d_{n+1}}{\longrightarrow} & P_n \\ & & f_n \downarrow \\ Q_{n+1} & \overset{d'_{n+1}}{\longrightarrow} & Q_n \end{array}.$$

Since $(Q_\bullet, d'_\bullet)$ is an exact complex we have $B_n(Q_\bullet) = Z_n(Q_\bullet)$.

Claim: $f_n(B_n(P_\bullet)) \subseteq Z_n(Q_\bullet)$.

Pf: Since the proof is standard I will leave it to the reader.

So, we have

$$\begin{array}{ccccc} P_{n+1} & \stackrel{d_{n+1}}{\longrightarrow} & B_n(P_\bullet) & & \\ & & f_n \downarrow & & \\ Q_{n+1} & \stackrel{d'_{n+1}}{\longrightarrow} & Z_n(Q_\bullet) & \to & 0 \end{array} .$$

Since P_{n+1} is projective, there is a map $f_{n+1} : P_{n+1} \to Q_{n+1}$ making this diagram commute. Since $B_n(P_\bullet) = Im(d_{n+1})$ it follows that

$$\begin{array}{ccccc} P_{n+1} & \stackrel{d_{n+1}}{\longrightarrow} & P_n & & \\ f_{n+1} \downarrow & & f_n \downarrow & & \\ Q_{n+1} & \stackrel{d'_{n+1}}{\longrightarrow} & Q_n & \to & 0 \end{array} .$$

also commutes, and that completes the first part of the theorem.

It remains to show the uniqueness of $f_\bullet$, up to chain homotopy.

So, suppose we have two chain maps lifting f. Call them $f_\bullet$ and $g_\bullet$ and let $h_\bullet = f_\bullet - g_\bullet$. We'll show that $h_\bullet$ is chain homotopic to the zero map i.e. there are $s_i : P_i \to Q_{i+1}$ for $(i \geq 0)$ and $s_{-1} : M \to Q_0$ such that $h_i = s_{i-1} d_i + d'_{i+1} s_i$.

Now we have:

$$\begin{array}{ccccccc} & & P_0 & \stackrel{\epsilon}{\longrightarrow} & M & \to & 0 \\ & & h_0 \downarrow & & f \downarrow & & \\ Q_1 & \stackrel{d_1}{\longrightarrow} & Q_0 & \stackrel{\eta}{\longrightarrow} & N & \to & 0 \end{array} .$$

We know $f\epsilon = \eta f_0 = \eta g_0$. Thus, $\eta h_0 = 0$ and so $Im(h_0) \subseteq Ker(\eta) = Im(d_1)$. So, we have:

$$\begin{array}{cccc} & P_0 & & \\ & \downarrow h_0 & & \\ Q_1 & \stackrel{d'_1}{\longrightarrow} & Im(d_1) & \to \; 0 \end{array} .$$

Since P_0 is projective there is a map $s_0 : P_0 \to Q_1$ such that $d'_1 s_0 = h_0$. If we set $s_{-1} = 0$-map then $h_0 = s_{-1}\eta_0 + d'_1 s_0$ and so we are on our way!

We construct the other maps, s_i, inductively. So, suppose we have s_i, $i < n$, that work well, i.e. we are OK up to h_{n-1}. In particular, $h_{n-1} = d'_n s_{n-1} + s_{n-2}d_{n-1}$. I.e. we have

$$\begin{array}{ccc} P_n & \xrightarrow{d_n} & P_{n-1} \\ h_n \downarrow & s_{n-1} \swarrow & h_{n-1} \downarrow \\ Q_n & \xrightarrow{d'_n} & Q_{n-1} \end{array}$$

Consider the map $h_n - s_{n-1}d_n$ from P_n to Q_n. Clearly,

$$d'_n(h_n - s_{n-1}d_n) = d'_n h_n - d'_n s_{n-1} d_n$$

$$= d'_n h_n - (h_{n-1} - s_{n-2}d_{n-1})d_n = d'_n h_n - h_{n-1}d_n = 0 \ .$$

(This last, since $h_\bullet$ is a chain map.)

Therefore, $Im(h_n - s_{n-1}d_n) \subseteq Ker(d'_n) = Im(d'_{n+1})$. So, we have

$$\begin{array}{cccc} & & P_n & \\ & & {\scriptstyle h_n - s_{n-1}d_n} \downarrow & \\ Q_{n+1} & \xrightarrow{d'_{n+1}} & Im(d'_{n+1}) & \to 0 \end{array}$$

so, by the fact that P_n is projective, there is a map s_n,

$$s_n : P_n \to Q_{n+1} \text{ such that } d'_{n+1}s_n = h_n - s_{n-1}d_n$$

as we wanted to show. $\square$

The next lemma shows how to construct a projective resolution for the middle module in a short exact sequence if we are given projective resolutions for the other two modules in the short exact sequences. Weibel referees to this result, graphically, as the Horseshoe Lemma.

Horseshoe Lemma: Let

$$0 \to A' \xrightarrow{i} A \xrightarrow{\pi} A'' \to 0$$

be a short exact sequence of R-modules and suppose that $(P'_\bullet, d'_\bullet, \epsilon')$ is a projective resolution of A' and that $(P''_\bullet, d''_\bullet, \epsilon'')$ is a projective resolution of A''.

Let $P_n = P'_n \oplus P''_n$. Then we can find maps, $d_\bullet$ and ϵ such that

1) $(P_\bullet = \{P_n\}, d_\bullet, \epsilon)$ is a projective resolution of A;
2) the exact sequences

$$0 \to P'_n \overset{i_{1,n}}{\to} P'_n \oplus P''_n \overset{\pi_{n,2}}{\longrightarrow} P''_n \to 0$$

come together so that: $i_\bullet$ is a chain map from the projective resolution of A' to that of A and $\pi_\bullet$ is a chain map from the projective resolution of A to that of A''.

Proof: We begin with the definition of ϵ. We have:

$$\begin{array}{ccccccccc}
0 & \to & P'_0 & \overset{i_{0,1}}{\longrightarrow} & P'_0 \oplus P''_0 & \overset{\pi_{0,2}}{\longrightarrow} & P''_0 & & \\
 & & \epsilon' \downarrow & & & & \downarrow \epsilon'' & & \\
0 & \to & A' & \overset{i}{\longrightarrow} & A & \overset{\pi}{\longrightarrow} & A'' & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

Since P''_0 is projective and π is onto there is a map

$$\delta_0 : P''_0 \to A \text{ such that } \pi \circ \delta_0 = \epsilon'' \ .$$

Now consider $\epsilon = (i \circ \epsilon', \delta_0)$. Then

$$\pi \circ \epsilon(a, b) = \pi i \epsilon'(a) + \pi\delta_0(b) = 0 + \pi\delta_0(b) \text{ while } \epsilon'' \circ \pi_{0,2}(a, b) = \epsilon''(b) \ .$$

Since $\pi\delta_0 = \epsilon''$ we get commutativity in this box. Commutativity in the other box is obvious.

So, we have constructed ϵ. Note that, by a piece of the Snake Lemma, ϵ is onto.

In fact, what we have (by the Snake Lemma) is the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Ker(\epsilon') & \to & Ker(\epsilon) & \to & Ker(\epsilon'') & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P_0' & \xrightarrow{i_{0,1}} & P_0' \oplus P_0'' & \xrightarrow{\pi_{0,2}} & P_0'' & \to & 0 \\
 & & \epsilon' \downarrow & & \downarrow \epsilon & & \downarrow \epsilon'' & & \\
0 & \to & A' & \xrightarrow{i} & A & \xrightarrow{\pi} & A'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

(where the exactness in the first row comes from the first zero in the last row!).

We thus also have the diagram,

$$
\begin{array}{ccccccccc}
 & & P_1' & \xrightarrow{i_{1,1}} & P_1' \oplus P_1'' & \xrightarrow{\pi_{1,2}} & P_1'' & & \\
 & & d_1' \downarrow & & & & \downarrow d_1'' & & \\
0 & \to & Ker(\epsilon') & \xrightarrow{i_{0,1}} & Ker(\epsilon) & \xrightarrow{\pi_{0,2}} & Ker(\epsilon'') & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}
$$

and now we can continue exactly as before where

$$d_1 : P_1' \oplus P_1'' \to Ker(\epsilon) \ , d_1 = (i_{0,1}d_1', \delta_1) \text{ and } \delta_1 : P_1'' \to Ker(\epsilon)$$

comes from the fact that P_1'' is projective.

The proof proceeds by induction. □

Roughly speaking the map we have constructed to make the middle complex is given as follows:

$$
\begin{array}{ccc}
P_n' & & P_{n-1}' \\
\oplus & \xrightarrow{A} & \oplus \\
P_n'' & & P_{n-1}''
\end{array}
$$

where

$$A = \begin{pmatrix} d'_n & \delta_n \\ 0 & d''_n \end{pmatrix} \text{ where } \delta_n : P''_n \to P'_{n-1} \ .$$

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I'd like to write the maps in the middle complex (given by the Horseshoe Lemma) in a more suggestive form.

Recall $\epsilon = (i \circ \epsilon', \delta_0)$ where $\epsilon' : P'_0 \to A'$, $i : A' \to A$ and $\delta_0 : P''_0 \to A$.

Since $\epsilon : P'_0 \oplus P''_0 \to A$, we can describe ϵ with a 1×2 matrix

$$\epsilon = [\epsilon' \quad \delta_0] \quad .$$

The map

$$\begin{matrix} P'_1 & & P'_0 \\ \oplus & \to & \oplus \\ P''_1 & & P''_0 \end{matrix}$$

was given by

$$d_1 \begin{bmatrix} a \\ b \end{bmatrix} = i_{0,1} d'_1(a) + \delta_1(b)$$

where $d'_1 : P'_1 \to P'_0$, $i_{0,1} : P'_0 \to P'_0 \oplus P''_0$ is inclusion in the first factor,

$$\pi_{0,2} : P'_0 \oplus P''_0 \to P''_0$$

is projection on the second factor, and $\delta_1 : P''_1 \to P'_0 \oplus P''_0$ satisfies $\pi_{0,2}\delta_1 = d''_1$.

To write this map in matrix form we need only see what happens to $\left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right]$.

Since $d_1 \left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right] = i_{0,1} d'_1(a)$, we get that the first column of the matrix representing d_1 is $\left[\begin{smallmatrix} d'_1 \\ 0 \end{smallmatrix}\right]$.

Now $d_1 \left[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right] = \delta_1(b) = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ where $s \in P'_0$ and $y \in P''_0$. We know that $\pi_{0,2}\delta_1 = d''_1$, and since $(\pi_{0,2}\delta_1)(b) = y$, we get that $y = d''_1(b)$. So, the second column of the matrix looks like $\left[\begin{smallmatrix} \lambda_1 \\ d''_1 \end{smallmatrix}\right]$ where $\lambda_1 : P''_1 \to P'_0$. I.e. we have that

$$d_1 = \begin{pmatrix} d'_1 & \lambda_1 \\ 0 & d''_1 \end{pmatrix}$$

in matrix form.

We can go on and do this at each level and we find

$$d_n = \begin{pmatrix} d'_n & \lambda_n \\ 0 & d''_n \end{pmatrix}$$

where $\lambda_n : P''_n \to P'_{n-1}$. The fact that $d_n d_{n+1} = 0$ for all n gives that

$$d'_n \lambda_{n+1} + \lambda_n d''_{n+1} = 0$$

for all n.

There is one other construction with chain complexes that I want to describe. It will also be useful in making projective resolutions of a module in a short exact sequence when we know the projective resolution of two of the modules.

Mapping Cones

Let $(A_\bullet, d_\bullet)$ and $(M_\bullet, d'_\bullet)$ be two chain complexes and let $\phi_\bullet : A_\bullet \to M_\bullet$ be a mapping of chain complexes.

Define $E_n = A_{n-1} \oplus M_n$ and Δ_n as follows:

$$\Delta_n(a_{n-1}, m_n) = (-d_{n-1}(a_{n-1}), \phi_{n-1}(a_{n-1}) + d'_n(m_n)) \ .$$

I.e. if we write elements of the direct sum as "column vectors" and use matrix notation, we have

$$\Delta_n = \begin{pmatrix} -d_{n-1} & 0 \\ \phi_{n-1} & d'_n \end{pmatrix} \ .$$

Clearly

$$\Delta_{n-1}\Delta_n = \begin{pmatrix} -d_{n-2} & 0 \\ \phi_{n-2} & d'_{n-1} \end{pmatrix} \begin{pmatrix} -d_{n-1} & 0 \\ \phi_{n-1} & d'_n \end{pmatrix}$$

$$= \begin{pmatrix} d_{n-2}d_{n-1} & 0 \\ -\phi_{n-2}d_{n-1} + d'_{n-1}\phi_{n-1} & d'_{n-1}d'_n \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ (*) & 0 \end{pmatrix}$$

where $(*) = 0$ since this is simply the statement that $\phi_\bullet$ is a chain map.

Definition: The chain complex (E_n, Δ_n) defined above is called the *mapping cone* over the map $\phi_\bullet : A_\bullet \to M_\bullet$ and is denoted $E(\phi_\bullet)$.

Proposition: The sequences

$$0 \to M_n \xrightarrow{i_n} E_n \xrightarrow{\pi_n} A_{n-1} \to 0$$

are exact for every n and

$$\begin{array}{ccccccccc} 0 & \to & M_n & \xrightarrow{i_n} & E_n & \xrightarrow{\pi_n} & A_{n-1} & \to & 0 \\ & & d'_n \downarrow & & \downarrow \Delta_n & & -d_{n-1} \downarrow & & \\ 0 & \to & M_{n-1} & \xrightarrow{i_{n-1}} & E_{n-1} & \xrightarrow{\pi_{n-1}} & A_{n-2} & \to & 0 \end{array}$$

commutes for each n.

So, if $\phi_\bullet : A_\bullet \to M_\bullet$ is a map of chain complexes and $i_\bullet$ and $\pi_\bullet$ are the chain maps described above, we have a short exact sequence of chain complexes

$$0 \to M_\bullet \xrightarrow{i_\bullet} E(\phi_\bullet) \xrightarrow{\pi_\bullet} A_\bullet(-1) \to 0$$

(where $[A_\bullet(-k)]_l = A_{l-k}$).

Proof: The exactness of each of the sequences is obvious since $E_n = M_n \oplus A_{n-1}$. As for the left-most square, let $m_n \in M_n$ then

$$m_n \to \begin{bmatrix} 0 \\ m_n \end{bmatrix} \to \begin{pmatrix} -d_{n-1} & 0 \\ \phi_{n-1} & d'_n \end{pmatrix} \begin{bmatrix} 0 \\ m_n \end{bmatrix} = \begin{bmatrix} 0 \\ d'_n(m_n) \end{bmatrix}$$

while going the other direction gives

$$m_n \to d'_n(m_n) \to \begin{bmatrix} 0 \\ d'_n(m_n) \end{bmatrix}$$

and this gives the required commutativity in that square.

As for the rightmost square, we have

$$\begin{bmatrix} a_{n-1} \\ m_n \end{bmatrix} \to a_{n-1} \to -d_{n-1}(a_{n-1}) \ .$$

On the other hand we have

$$\begin{bmatrix} a_{n-1} \\ m_n \end{bmatrix} \to \begin{pmatrix} -d_{n-1} & 0 \\ \phi_{n-1} & d'_n \end{pmatrix} \begin{bmatrix} a_{n-1} \\ m_n \end{bmatrix} = \begin{bmatrix} -d_{n-1}(a_{n-1}) \\ \phi_{n-1}(a_{n-1}) + d'_n(m_n) \end{bmatrix} \to -d_{n-1}(a_{n-1}) \ ,$$

so that square commutes as well.

The remaining statements of the proposition are immediate corollaries of the remarks above. $\square$

Once we observe that if $(A_\bullet, d_\bullet)$ is a chain complex with homology modules $H_n(A_\bullet)$ then the chain complex $(A_\bullet, -d_\bullet)$ has exactly the same homology modules, it is clear from the Proposition above, that we get the following long exact sequence of homology involving the mapping cone. Namely:

$$\cdots \to H_n(A_\bullet) \xrightarrow{\partial_n} H_n(M_\bullet) \xrightarrow{i_\bullet} H_n(E_\bullet) \xrightarrow{\pi_\bullet} H_{n-1}(A_\bullet) \xrightarrow{\partial_{n-1}} H_{n-1}(M_\bullet) \to \cdots$$

The interesting thing about this sequence is the following:

Claim: The maps $\partial_{n-1} = H_{n-1}(\phi_\bullet)$.

Pf: Recall how the connecting morphisms are defined. Let $a_{n-1} \in Z_{n-1}(A_\bullet)$, then

$$a_{n-1} = \pi_n \begin{bmatrix} a_{n-1} \\ 0 \end{bmatrix}$$

and

$$\Delta_n \begin{bmatrix} a_{n-1} \\ 0 \end{bmatrix} = \begin{pmatrix} -d_{n-1} & 0 \\ \phi_{n-1} & d'_n \end{pmatrix} \begin{bmatrix} a_{n-1} \\ 0 \end{bmatrix} = \begin{bmatrix} -d_{n-1}(a_{n-1}) \\ \phi_{n-1}(a_{n-1}) \end{bmatrix} = \begin{bmatrix} 0 \\ \phi_{n-1}(a_{n-1}) \end{bmatrix}$$

since $a_{n-1} \in Z_{n-1}(A_\bullet)$.

Now,

$$i_{n-1}(\phi_{n-1}(a_{n-1})) = \begin{bmatrix} 0 \\ \phi_{n-1}(a_{n-1}) \end{bmatrix}$$

and the connecting morphism takes $a_{n-1} \to \overline{\phi_{n-1}(a_{n-1})}$, which is what $H_{n-1}(\phi_\bullet)$ does also. That observation completes the proof. $\square$

We can use the notion of the mapping cone to "fill-in" the resolution of a module in a short exact sequence if we know the resolutions of the other two modules, as follow.

Let

$$0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$$

be a short exact sequence of R-modules. Let $(P'_\bullet, d'_\bullet, \epsilon')$ be a projective resolution of M' and let $(P_\bullet, d_\bullet, \epsilon)$ be a projective resolution of M. Suppose further that $\alpha_\bullet : P'_\bullet \to P_\bullet$ is any lift of α.

Proposition: Then $E(\alpha_\bullet)$ (the mapping cone over α) is a projective resolution of M'' and the inclusion maps, $i_n : P_n \to P'_{n-1} \oplus P_n$ form a chain map which lifts β.

Proof: We have the following diagram

$$\begin{array}{ccccccccc}
 & & P'_0 & \xrightarrow{\alpha_0} & P_0 & \xrightarrow{id} & P_0 & & \\
 & & \epsilon' \downarrow & & \epsilon \downarrow & & \beta\epsilon \downarrow & & \\
0 & \to & M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

which starts the appropriate resolutions and maps. So, let's consider the next step.

We have

$$\begin{array}{ccccc}
P'_1 & \xrightarrow{\alpha_1} & P_1 & \xrightarrow{i_2} & P'_0 \oplus P_1 \\
d'_1 \downarrow & & d_1 \downarrow & & \Delta_1 \downarrow \\
P'_0 & \xrightarrow{\alpha_0} & P_0 & \xrightarrow{id} & 0 \oplus P_0
\end{array}$$

where $\Delta_1 = \begin{bmatrix} -d'_0 & 0 \\ \alpha_1 & d_1 \end{bmatrix}$.

It is easy to see that the diagrams above commute: the left side in view of the assumption that $\alpha_\bullet$ is a chain map; the right side because, going one way we have:

$$p_1 \to \begin{bmatrix} 0 \\ p_1 \end{bmatrix} \to \Delta_1 \begin{bmatrix} 0 \\ p_1 \end{bmatrix} = \begin{bmatrix} 0 \\ d_1(p_1) \end{bmatrix}$$

while going the other way gives

$$p_1 \to d_1(p_1) \to \begin{bmatrix} 0 \\ d_1(p_1) \end{bmatrix}.$$

By induction, at the n-th step we have:

$$\begin{array}{ccccc} P'_n & \xrightarrow{\alpha_n} & P_n & \xrightarrow{i_n} & P'_{n-1} \oplus P_n \\ d'_n \downarrow & & d_n \downarrow & & \Delta_n \downarrow \\ P'_{n-1} & \xrightarrow{\alpha_{n-1}} & P_{n-1} & \xrightarrow{i_{n-1}} & P'_{n-2} \oplus P_{n-1} \end{array}$$

where $\Delta_n = \begin{bmatrix} -d'_{n-1} & 0 \\ \alpha_{n-1} & d_n \end{bmatrix}$.

As far as commutativity in these squares is concerned, the left side comes from the fact that $\alpha_\bullet$ is a chain map and the right side because, going one way gives:

$$p_n \to \begin{bmatrix} 0 \\ p_n \end{bmatrix} \to \begin{pmatrix} -d'_{n-1} & 0 \\ \alpha_{n-1} & d_n \end{pmatrix} \begin{bmatrix} 0 \\ p_n \end{bmatrix} = \begin{bmatrix} 0 \\ d_n(p_n) \end{bmatrix}$$

while going the other way gives

$$p_n \to d_n(p_n) \to \begin{bmatrix} 0 \\ d_n(p_n) \end{bmatrix}$$

as we wanted to show.

Finally, since we have a short exact sequence of chain complexes and the first two are exact, then so is the third. So, we have, indeed, constructed a projective resolution of the the module M''. $\square$

We leave "projective resolutions" for a while and return to our discussion of universal δ-functors.

Exercise 16A:(10) Show that a right exact covariant functor is an additive functor (from R-mod to R-mod). Do the same for a left exact covariant functor. (Check, but do not hand in, that the same is true for contravariant functors.)

Let $F : R\text{-mod} \to R\text{-mod}$ be a covariant right exact functor.

We will define a family of associated functors to F; these are called the "left " *derived functors* of F, and denoted L_iF for $(i \geq 0)$. These functors are defined as follows:

let A be an R-module. Choose, once and for all, a particular projective resolution of A and call it $(P_\bullet, d_\bullet, \epsilon)$. Then we define

$$L_i F(A) := H_i(F(P_\bullet)) \ .$$

Remarks:

1). Since

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$$

is exact and F is right exact, we have that

$$F(P_1) \xrightarrow{F(d_1)} F(P_0) \xrightarrow{F(\epsilon)} F(A) \to 0 \ . \tag{$\dagger$}$$

is also exact.

Now, the complex $F(P_\bullet)$ ends

$$F(P_1) \xrightarrow{F(d_1)} F(P_0) \to 0$$

and the 0-th homology module of this complex is $F(P_0)/Im(F(d_1))$. But, by ($\dagger$), this is exactly $F(A)$. Thus, $(L_0 F)(A) \simeq F(A)$.

2). Note also that if $f : A \to B$ is a homomorphism of R-modules and if $(P_\bullet, d_\bullet, \epsilon)$ is the chosen projective resolution of A and if $(Q_\bullet, d'_\bullet, \epsilon')$ is the chosen projective resolution of B, then we saw how to lift f to a morphism of complexes $f_\bullet : P_\bullet \to Q_\bullet$.

If we apply F to this picture, we obtain two complexes $F(P_\bullet)$ and $F(Q_\bullet)$ and a morphism $F(f_\bullet)$ of those complexes. Since the $L_i F$ are just the homology modules of these complexes, they induce maps $L_i F(f) : L_i F(A) \to L_i F(B)$. It is easy to see that composition works well (first lift f, then lift g and then lift gf and use the comparison theorem). Also, the identity map lifts to the identity chain map and the induced maps on homology are the identity. So, we do indeed have a family of functors. As we saw earlier, these functors are also additive (they are homology functors and we showed earlier that those were additive.)

The aim is to show that these functors are a (covariant) universal homological δ-functor.

Lemma 1: The objects $(L_i F)(A)$ are well-defined, up to natural isomorphism.

I.e. if $\widetilde{P} = (P_\bullet, d_\bullet, \epsilon)$ and $\widetilde{P'} = (P'_\bullet, d'_\bullet, \epsilon')$ are two projective resolutions of A then there is a natural isomorphism

$$H_i(F(P_\bullet)) \simeq H_i(F(P'_\bullet)) \ . \qquad (*)$$

Put another way, if we let $L_i F$ denote the functors obtained from the first choice of a projective resolution for every R-module A and let $L'_i F$ be the functors obtained by making a different choice of projective resolutions for each R-module A, then the isomorphisms given above $(*)$, define a natural isomorphism of the functors $L_i F$ and $L'_i F$.

Proof: By the Comparison Theorem, there is a chain map $f_\bullet : \widetilde{P} \to \widetilde{P'}$ which lifts the identity map $id_A : A \to A$. This gives maps

$$H_i(F(f_\bullet)) : H_i(F(P_\bullet)) \to H_i(F(P'_\bullet)) \ .$$

Since any other chain map lifting id_A is chain homotopic to $f_\bullet$, and since F is additive, the maps between the chain complexes $F(P_\bullet)$ and $F(P'_\bullet)$ are also chain homotopic and so the maps $H_i(F(f_\bullet))$ are independent of the lifting of id_A.

But we can also go the other way, and get $g_\bullet : \widetilde{P'} \to \widetilde{P}$ lifting $\mathrm{id}_A : A \to A$, as before, giving maps

$$H_i(F(g_i)) : H_i(F(P'_\bullet)) \to H_i(F(P_\bullet))$$

which are independent of the lifting.

Composing these chain maps in one direction, gives a chain map $\widetilde{P} \to \widetilde{P}$ which lifts id_A. Since this is independent of the lifting and since $\mathrm{id}_{\widetilde{P}}$ is a lifting, we get that

$$H_i(F(g_\bullet))H_i(F(f_\bullet)) = id \ .$$

Similarly,

$$H_i(F(f_\bullet))H_i(F(g_\bullet)) = id \ .$$

So, the maps are all isomorphisms.

To show that these isomorphisms give a natural transformation of functors we need to show that, given $f : A \to B$, and the isomorphisms given above, then the diagram below commutes.

$$\begin{array}{ccccc} A & & (L_iF)(A) & \xrightarrow{\simeq} & (L'_iF)(A) \\ f\downarrow & & {\scriptstyle (L_iF)(f)}\downarrow & & \downarrow{\scriptstyle (L'_iF)(f)} \\ B & & (L_iF)(B) & \xrightarrow{\simeq} & (L'_iF)(B) \end{array} .$$

I'll leave the details to you to work out, but this follows using the Comparison Theorem because the chain maps

$$\begin{array}{ccc} P_\bullet & \to & A \\ & & \downarrow f \\ Q_\bullet & \to & B \\ & & \downarrow id \\ Q'_\bullet & \to & B \end{array} \qquad\qquad \begin{array}{ccc} P_\bullet & \to & A \\ & & \downarrow id \\ P'_\bullet & \to & A \\ & & \downarrow f \\ Q'_\bullet & \to & B \end{array}$$

(where $P_\bullet$ and $P'_\bullet$ are the two resolutions of A and $Q_\bullet$ and $Q'_\bullet$ are the two resolutions of B) between $P_\bullet$ and $Q'_\bullet$ which lift f always induce the same map on homology, independent of the lifting. Going around the square one way, and then the other, correspond to this. □

Corollary: If A is a projective R-module then

$$(L_iF)(A) = 0 \text{ for all } i > 0 .$$

Proof: This follows immediately from the fact that

$$\cdots 0 \to A \xrightarrow{id} A \to 0$$

is a projective resolution of A. □

So, we have seen that if we choose a fixed projective resolution of each object in R-mod, and a right exact functor, F from R-mod to itself, we can define a collection of additive functors, the L_iF, also from R-mod to itself. The functors we've seen, are somewhat independent of the resolution we choose of the objects in R-mod, in the sense that, for a different choice of projective resolution for the objects of R-mod, we get a different family of functors, but the two families of functors are naturally isomorphic. So, we can stick with one resolution for each object, and work with it.

Theorem: The derived functors $\{L_iF\}_{i\geq 0}$, form a (covariant) homological δ-functor.

Proof: We have already noted that these are a family of additive functors, so to prove the theorem we have to show two things:

i) Given a short exact sequence of R-modules, we get a long exact sequence of homology, with connecting morphisms, δ_n, for each n.

ii) The δ_n are "natural", i.e. given two short exact sequences and maps between them that give rise to commutative diagrams, then the resulting two long exact sequences of homology (with the given connecting morphisms) are linked by commutative diagrams.

So, we begin with *i*), i.e. let

$$0 \to A' \to A \to A'' \to 0$$

be a short exact sequence of R-modules. Let $P'_\bullet$ be a projective resolution of A' and $P''_\bullet$ be a projective resolution of A''. By the Horseshoe Lemma there is a projective resolution $P_\bullet$ of A fitting into a short exact sequence of complexes

$$0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$$

where, for each n we have

$$0 \to P'_n \to P'_n \oplus P''_n \to P''_n \to 0$$

each of which is a split exact sequence.

Since the functor F is additive, we have

$$0 \to F(P'_n) \to F(P'_n) \oplus F(P''_n) \to F(P''_n) \to 0$$

is also split exact. I.e.

$$o \to F(P'_\bullet) \to F(P_\bullet) \to F(P''_\bullet) \to 0$$

is a short exact sequence of chain complexes. Therefore, by the theorem about the long exact sequence of homology we get:

$$\cdots \xrightarrow{\delta_{n+1}} (L_nF)(A') \to (L_nF)(A) \to (L_nF)(A'') \xrightarrow{\delta_n} \cdots$$

and that proves i).

We now need to show the *naturality* of the connecting morphisms.

So, suppose we are given a homomorphism of short exact sequences of R-modules, i.e. we have a diagram

$$\begin{array}{ccccccccc} 0 & \to & A' & \to & A & \to & A'' & \to & 0 \\ & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f''}\downarrow & & \\ 0 & \to & B' & \to & B & \to & B'' & \to & 0 \end{array}$$

and let $P'_\bullet$, $P''_\bullet$, $Q'_\bullet$, $Q''_\bullet$ be projective resolutions of A', A'', B', B'' respectively. Let $P_\bullet$ and $Q_\bullet$ be the projective resolutions arising, by the Horseshoe Lemma, for A and B respectively. So, these are what we used above to define the connecting morphisms for each of the exact sequences above.

I.e. we get two short exact sequences of chain complexes

$$0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$$

and

$$0 \to Q'_\bullet \to Q_\bullet \to Q''_\bullet \to 0 \ .$$

Now, we can lift f' to a chain map $F'_\bullet : P'_\bullet \to Q'_\bullet$ and we can also lift f'' to a chain map $F''_\bullet : P''_\bullet \to Q''_\bullet$. This brings us to the crux of the matter.

Claim: There is a chain map $F_\bullet : P_\bullet \to Q_\bullet$ lifting f AND such that

$$\begin{array}{ccccccccc} 0 & \to & P'_\bullet & \to & P_\bullet & \to & P''_\bullet & \to & 0 \\ & & {\scriptstyle F'_\bullet}\downarrow & & {\scriptstyle F_\bullet}\downarrow & & {\scriptstyle F''_\bullet}\downarrow & & \\ 0 & \to & Q'_\bullet & \to & Q_\bullet & \to & Q''_\bullet & \to & 0 \end{array}$$

is a commutative diagram of short exact sequences of chain complexes of projective modules.

Remark: Once we've proved this Claim, we are finished with the theorem. This is so because the naturality of the connecting morphisms for the L_iF is precisely the naturality of the connecting morphism in the long exact homology sequence above, which we proved earlier! So, we need to find the middle chain map making the diagram above commute.

For each n, we have

$$\begin{array}{ccccccccc} 0 & \to & P'_n & \to & P'_n \oplus P''_n & \to & P''_n & \to & 0 \\ & & {\scriptstyle F'_n} \downarrow & & & & {\scriptstyle F''_n} \downarrow & & \\ 0 & \to & Q'_n & \to & Q'_n \oplus Q''_n & \to & Q''_n & \to & 0 \end{array}$$

and we want F_n, in the middle, to make the diagrams commute, and be part of a lifting for f.

We propose a certain form for F_n, namely, let

$$F_n = \begin{pmatrix} F'_n & \gamma_n \\ 0 & F''_n \end{pmatrix} : \begin{array}{c} P'_n \\ \oplus \\ P''_n \end{array} \to \begin{array}{c} Q'_n \\ \oplus \\ Q''_n \end{array}$$

where $\gamma_n : P'_n \to Q''_n$.

Let's check the commutativity of the squares: as for the leftmost square, we have

$$a \to \begin{bmatrix} a \\ 0 \end{bmatrix} \to \begin{bmatrix} F'_n(a) \\ 0 \end{bmatrix}$$

one way, and

$$a \to F'_n(a) \to \begin{bmatrix} F'_n(a) \\ 0 \end{bmatrix}$$

so that square commutes, regardless of any choice for γ_n.

As for the righthand square, we have

$$\begin{bmatrix} a \\ b \end{bmatrix} \to b \to F''_n(b)$$

going one way, and

$$\begin{bmatrix} a \\ b \end{bmatrix} \to \begin{pmatrix} F_n' & \gamma_n \\ 0 & F_n'' \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} F_n'(a) + \gamma_n(b) \\ F_n''(b) \end{bmatrix} \to F_n''(b)$$

again regardless of the choice of γ_n.

So, the only thing we have to worry about in our choices here are that the chain map $F_\bullet$ is a lifting of f.

In order to lift f we must have

$$\begin{array}{ccc} P_0' \oplus P_0'' & \xrightarrow{F_0} & Q_0' \oplus Q_0'' \\ \epsilon \downarrow & & \eta \downarrow \\ A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

i.e.

$$f \circ \epsilon \begin{bmatrix} a \\ b \end{bmatrix} = \eta \begin{pmatrix} F_0' & \gamma_0 \\ 0 & F_0'' \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \eta \begin{bmatrix} F_0'(a) + \gamma_0(b) \\ F_0''(b) \end{bmatrix}$$

Thus, what we want is that

$$f \circ \epsilon - \eta \begin{pmatrix} F_0' & \gamma_0 \\ 0 & F_0'' \end{pmatrix} \qquad (*)$$

be the zero map from $P_0' \oplus P_0'' \to B$.

We must recall, however, how η and ϵ were defined, by the Horseshoe Lemma!

So, if we name the maps in the "B" sequence as

$$0 \to B' \xrightarrow{i_2} B \xrightarrow{\pi_2} B'' \to 0$$

and call η'' the augmentation map for the projective resolution of B'', and η' the augmentation map for the projective resolution of B' then

$$\eta = (i_2\eta', \delta_{B,0}) \text{ where } \delta_{B,0} : Q_0'' \to B \text{ is such that } \pi_2\delta_{B,0} = \eta'' \ .$$

Similarly, if we name the maps in the "A" sequence as

$$0 \to A' \xrightarrow{i_1} A \xrightarrow{\pi_1} A'' \to 0$$

and call ϵ'' the augmentation map for the projective resolution of A'', and ϵ' the augmentation map for the projective resolution of A' then

$$\epsilon = (i_1\epsilon', \delta_{A,0}) \text{ where } \delta_{A,0} : P_0'' \to A \text{ is such that } \pi_1\delta_{A,0} = \epsilon'' \ .$$

Setting $(*) = 0$, which is what we want, means

$$f \circ \epsilon \begin{bmatrix} a \\ b \end{bmatrix} = f\big(i_1\epsilon'(a) + \delta_{A,0}(b)\big) = \eta \begin{bmatrix} F_0'(a) + \gamma_0(b) \\ F_0''(b) \end{bmatrix} = i_2\eta'(F_0'(a) + \gamma_0(b)) + \delta_{B,0}F_0''(b) \ .$$

I.e. we want γ_0 such that for $a \in P_0'$ and $b \in P_0''$,

$$f i_1 \epsilon'(a) + f\delta_{A,0}(b) = i_2\eta' F_0'(a) + i_2\eta'\gamma_0(b) + \delta_{B,0}F_0''(b) \qquad (**)$$

We do have the following commutative diagram,

$$\begin{tikzcd}
P_0' \arrow[d, "F_0'"'] \arrow[dr, "\epsilon'"] & & \\
Q_0' \arrow[dr, "\eta'"'] & A' \arrow[r, "i_1"] \arrow[d, "f'"'] & A \arrow[d, "f"'] \\
& B' \arrow[r, "i_2"] & B
\end{tikzcd}$$

so

$$f i_1 \epsilon'(a) = i_2\eta' F_0'(a) \text{ for any } a \in P_0' \ .$$

Thus, we may cancel those terms from $(**)$ above, and it remains to show that we can find γ_0 such that

$$i_2\eta'\gamma_0 = f\delta_{A,0} - \delta_{B,0}F_0'' \ . \qquad (***)$$

However, we also have commutative diagrams

$$\begin{tikzcd}
& & P_0'' \arrow[dl, "\epsilon''"'] \arrow[d, "F_0''"] \\
A \arrow[r, "\pi_1"] \arrow[d, "f"'] & A'' \arrow[d, "f''"'] & Q_0'' \arrow[dl, "\eta''"] \\
B \arrow[r, "\pi_2"] & B'' &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& P_0'' \arrow[dl, "\delta_{A,0}"'] \arrow[d, "F_0''"] \\
A \arrow[d, "f"'] & Q_0'' \arrow[dl, "\delta_{B,0}"] \\
B &
\end{tikzcd}$$

and so

$$\pi_2(f\delta_{A,0} - \delta_{B,0}F_0'') = f''\pi_1\delta_{A,0} - \pi_2\delta_{B,0}F_0'' = f''\epsilon'' - \eta''F_0'' = 0 \ .$$

i.e. $Im(f\delta_{A,0} - \delta_{B,0}F_0'') \subseteq Ker(\pi_2) = Im(i_2)$.

So, we have the following diagram

$$\begin{tikzcd}
& P_0'' \arrow[d, "f\delta_{A,0}-\delta_{B,0}F_0''"] & \\
B' \arrow[r, "i_2"] & Im(i_2) \arrow[r] & 0
\end{tikzcd}$$

and so we can find a homomorphism $\beta : P_0'' \to B'$ such that $i_2\beta = f\delta_{A,0} - \delta_{B,0}F_0''$. If we substitute this into $(***)$ above, we see that we want γ_0 such that

$$i_2\eta'\gamma_0 = i_2\beta \text{ but, since } i_2 \text{ is } 1-1, \text{ we want } \eta'\gamma_0 = \beta \ .$$

But, from the diagram,

$$\begin{tikzcd}
& P_0'' \arrow[d, "\beta"] & \\
Q_0' \arrow[r, "\eta'"] & B' \arrow[r] & 0
\end{tikzcd}$$

we see that γ_0 can be chosen to be any map from P_0'' to P_0', which makes this diagram commute.

Thus, the chain map begins, i.e. we have

$$\begin{array}{ccccccccc}
\cdots & \to & \begin{matrix} P_1' \\ \oplus \\ P_1'' \end{matrix} & \xrightarrow{d_1} & \begin{matrix} P_0' \\ \oplus \\ P_0'' \end{matrix} & \xrightarrow{\epsilon} & A & \to & 0 \\
 & & & & F_0 \downarrow & & f \downarrow & & \\
\cdots & \to & \begin{matrix} Q_1' \\ \oplus \\ Q_1'' \end{matrix} & \xrightarrow{\widetilde{d}_1} & \begin{matrix} Q_0' \\ \oplus \\ Q_0'' \end{matrix} & \xrightarrow{\eta} & B & \to & 0
\end{array}$$

and our next step is to define F_1.

To do that we shall have to have

$$\widetilde{d}_1 F_1 = F_0 d_1$$

where, recall, $d'_\bullet$ and $d''_\bullet$ are the chain maps of $P'_\bullet$ and $P''_\bullet$ respectively, while $\widetilde{d'_\bullet}$ and $\widetilde{d''_\bullet}$ are the chain maps of $Q'_\bullet$ and $Q''_\bullet$ respectively and

$$d_1 = \begin{pmatrix} d_1' & \lambda_1 \\ 0 & d_1'' \end{pmatrix} \quad \text{and} \quad \widetilde{d}_1 = \begin{pmatrix} \widetilde{d_1'} & \widetilde{\lambda_1} \\ 0 & \widetilde{d_1''} \end{pmatrix}$$

So, if we want

$$F_1 = \begin{pmatrix} F_1' & \gamma_1 \\ 0 & F_1'' \end{pmatrix}$$

then we must choose γ_1 so that

$$\begin{pmatrix} \widetilde{d_1'} & \widetilde{\lambda_1} \\ 0 & \widetilde{d_1''} \end{pmatrix} \begin{pmatrix} F_1' & \gamma_1 \\ 0 & F_1'' \end{pmatrix} - \begin{pmatrix} F_0' & \gamma_0 \\ 0 & F_0'' \end{pmatrix} \begin{pmatrix} d_1' & \lambda_1 \\ 0 & d_1'' \end{pmatrix} = 0$$

i.e.

$$\begin{pmatrix} \widetilde{d_1'} F_1' - F_0' d_1' & (\widetilde{d_1'}\gamma_1 + \widetilde{\lambda_1} F_1'') - (F_0'\lambda_1 + \gamma_0 d_1'') \\ 0 & \widetilde{d_1''} F_1'' - F_0'' d_1'' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

But, since $F'_\bullet$ is a chain map from $(P'_\bullet, d'_\bullet)$ to $(Q'_\bullet, \widetilde{d'_\bullet}$, we automatically have $\widetilde{d'_1}F'_1 = F'_0 d'_1$. Similarly, since $F''_\bullet$ is a chain map from $(P''_\bullet, d''_\bullet)$ to $(Q''_\bullet, \widetilde{d''_\bullet})$ we also get $\widetilde{d''_1}F''_1 - F''_0 d''_1 = 0$.

So, the only problem is to construct γ_1 so that

$$\widetilde{d'_1}\gamma_1 = -\widetilde{\lambda_1}F''_1 + F'_0\lambda_1 + \gamma_0 d''_1 \ .$$

But, consider

$$\eta'(-\widetilde{\lambda_1}F''_1 + F'_0\lambda_1 + \gamma_0 d''_1) \ .$$

I would like to show this composition is $= 0$. To do that it will be enough, since i_2 is $1-1$, to show that

$$i_2(\eta'(-\widetilde{\lambda_1}F''_1 + F'_0\lambda_1 + \gamma_0 d''_1)) = 0$$

But, recall that

$$(\, i_1\epsilon' \quad \delta_0 \,)\begin{pmatrix} d'_1 & \lambda_1 \\ 0 & d''_1 \end{pmatrix} = (\, i_1\epsilon' d'_1 \quad i_1\epsilon'\lambda_1 + \delta_0 d''_1 \,) = (\, 0 \quad 0 \,) \tag{$\dagger$}$$

and

$$(\, i_2\eta' \quad \delta'_0 \,)\begin{pmatrix} \widetilde{d'_1} & \widetilde{\lambda_1} \\ 0 & \widetilde{d''_1} \end{pmatrix} = \left(\, i_2\eta'\widetilde{d'_1} \quad i_2\eta'\widetilde{\lambda_1} + \delta'_0\widetilde{d''_1} \,\right) = (\, 0 \quad 0 \,) \tag{$\dagger\dagger$}$$

and, by the commutativity of the earlier square,

$$(\, i_2\eta' \quad \delta'_0 \,)\begin{pmatrix} F'_0 & \gamma_0 \\ 0 & F''_0 \end{pmatrix} = f\,(\, i_1\epsilon' \quad \delta_0 \,)$$

i.e.

$$(\, i_2\eta' F'_0 \quad i_2\eta'\gamma_0 + \delta'_0 F''_0 \,) = (\, f i_1\epsilon' \quad f\delta_0 \,) \tag{$\dagger\dagger\dagger$}$$

Thus,

$$i_2(\eta'(-\widetilde{\lambda_1}F''_1 + F'_0\lambda_1 + \gamma_0 d''_1)) = (-i_2\eta'\widetilde{\lambda_1})F''_1 + i_2\eta' F'_0\lambda_1 + (i_2\eta'\gamma_0)d''_1$$

which, by ($\dagger\dagger$) and ($\dagger\dagger\dagger$) gives

$$(\delta'_0\widetilde{d''_1})F''_1 + i_2\eta' F'_0\lambda_1 + (-\delta'_0 F''_0 + f\delta_0)d''_1 \ .$$

If we cancel like terms and use the fact that $F''_\bullet$ is a chain map, we are left with

$$= i_2\eta' F'_0\lambda_1 + f(\delta_0 d''_1)$$

which, by (†) is

$$= i_2\eta' F'_0\lambda_1 - f(i_1\epsilon'\lambda_1) = (i_2\eta' F'_0 - f i_1\epsilon')\lambda_1$$

and the last parenthetical expression is 0 by (†††).

Thus, $g_1 = -\widetilde{\lambda_1}F''_1 + F'_0\lambda_1 + \gamma_0 d''_1 : P''_1 \to Q'_0$ satisfies $\eta' \circ g_1 = 0$ and so $Im(g_1) \subseteq Ker(\eta') = Im(\widetilde{d'_1})$, i.e. we have a diagram

$$\begin{array}{ccccc} & & P''_1 & & \\ & & {}_{g_1}\downarrow & & \\ Q'_1 & \xrightarrow{\widetilde{d'_1}} & Im(\widetilde{d'_1}) & \to & 0 \end{array}$$

and so there is a map $\gamma_1 : P''_1 \to Q'_1$ such that $\widetilde{d'_1}\gamma_1 = g_1$ and so we are done in this case as well.

We proceed by induction and suppose we have constructed the appropriate maps $F_0,\ F_1, \ldots, F_{n-1}$ and we want to construct F_n. Proceeding as before we see that it is enough to construct $\gamma_n : P''_n \to Q'_n$ such that

$$\widetilde{d'_n}\gamma_n = -\widetilde{\lambda_n}F''_n + F'_{n-1}\lambda_n + \gamma_{n-1}d''_n = g_n\ .$$

If we use the inductive formula for $\widetilde{d'_{n-1}}\gamma_{n-1}$, the fact that $F''_\bullet$ is a chain map, $F'_\bullet$ is a chain map and

$$\widetilde{\lambda_{n-1}}\widetilde{d'_n} + \widetilde{d'_{n-1}}\widetilde{\lambda_n} = 0 = d'_{n-1}\lambda_n + \lambda_{n-1}d''_n$$

(coming from the matrix representation of the maps constructed by the Horseshoe Lemma), we find that:

$$\widetilde{d'_{n-1}}g_n = 0 \text{ i.e. } Im(g_n) \subseteq Ker(\widetilde{d'_{n-1}}) = Im(\widetilde{d'_n})$$

(the latter equality being true since $Q'_\bullet$ is a resolution of B'). Thus, we have the diagram

$$\begin{array}{ccccc} & & P''_n & & \\ & & {}_{g_n}\downarrow & & \\ Q'_n & \xrightarrow{\widetilde{d'_n}} & Im(\widetilde{d'_n}) & \to & 0 \end{array}$$

which can be completed with a map γ_n making the diagram commute and providing us with the map we need.

That completes the proof of this theorem. □

The only thing left, in this circle of ideas, is to show that the derived functors are, in fact UNIVERSAL! That is the content of the next theorem.

Theorem: Let F be a covariant right exact functor from R-mod to R-mod. The derived functors, $L_iF,\ i \geq 0$ form a universal homological δ-functor.

Proof: Let $\{T_n\}$ be another homological δ-functor and let $\phi_0 : T_0 \to F$ be a natural transformation of functors.

We need to show that ϕ_0 lifts to a morphism of homological δ-functors, i.e. a family of natural transformations $\phi_n : T_n \to L_nF$ which commute with the δ's of each family (as we made precise earlier).

We prove this inductively, and suppose that we have, inductively, defined natural transformations $\phi_i : T_i \to L_iF$, for $0 \leq i < n$ and these natural transformations commute with the appropriate δ's.

The construction of ϕ_n :

Pick $A \in R$-mod. We need, first, to define $\phi_n : T_n(A) \to (L_nF)(A)$.

We do this in what appears, at first, to be a very "unnatural" way! We pick, once and for all, a projective R-module mapping onto A (and we do this for every R-module A). Having done this we get a short exact sequence

$$0 \to K \to P \to A \to 0\ .$$

Since $\{L_iF\}$ and $\{T_i\}$ are both homological δ-functors, and we have natural transformations, up to degree $n-1$, we get (from the long exact homology sequences, and the fact that $(L_iF)(P) = 0$ for all $i > 0$),

$$\begin{array}{ccccccc} & & T_n(A) & \xrightarrow{\delta_n} & T_{n-1}(K) & \to & T_{n-1}(P) \\ & & & & \phi_{n-1}\downarrow & & \phi_{n-1}\downarrow \\ 0 & \to & (L_nF)(A) & \xrightarrow{\delta_n} & (L_{n-1}F)(K) & \to & (L_{n-1}F)(P) \end{array}\ .$$

Let $x \in T_n(A)$, then $\phi_{n-1}\delta_n(x) \in (L_{n-1}F)(K)$ is in the image of δ_n (the right hand square commutes and both top and bottom lines are complexes). Since the δ_n of the bottom row is $1-1$ we can find $y \in (L_nF)(A)$ such that $\delta_n(y) = \phi_{n-1}\delta_n(x)$.

We define: $\phi_n(x) = y$ and note: this is well-defined, is an R-module homomorphism and makes the appropriate square commute (by definition!).

We need to show that the maps so defined (for each A) are a natural transformation of functors. So, let $f : A' \to A$ be a homomorphism. Then, we get a diagram

$$\begin{array}{ccccccccc} 0 & \to & K' & \to & P' & \to & A' & \to & 0 \\ & & & & & & f\downarrow & & \\ 0 & \to & K & \to & P & \to & A & \to & 0 \end{array}$$

Since P' is projective we can lift f to g, and that, in turn induces h, as in the diagram below

$$\begin{array}{ccccccccc} 0 & \to & K' & \to & P' & \to & A' & \to & 0 \\ & & h\downarrow & & g\downarrow & & f\downarrow & & \\ 0 & \to & K & \to & P & \to & A & \to & 0 \end{array}$$

On the $\{T_i\}$ level, since they are homological δ-functors, we have

$$\begin{array}{ccccc} T_n(A') & & \xrightarrow{T_n(f)} & & T_n(A) \\ & \delta\searrow & & \swarrow\delta & \\ & T_{n-1}(K') & \xrightarrow{T_{n-1}(h)} & T_{n-1}(K) & \end{array}$$

Similarly, on the L_iF level we have:

$$\begin{array}{ccccc} & (L_{n-1}F)(K') & \xrightarrow{(L_{n-1}F)(h)} & (L_{n-1}F)(K) & \\ \delta\nearrow & & & & \delta\nwarrow \\ (L_nF)(A') & & \xrightarrow{(L_nF)(f)} & & (L_nF)(A) \end{array}$$

Also, since ϕ_{n-1} is a natural transformation of functors, we have the following commutative diagram:

$$\begin{tikzcd}
T_{n-1}(K') \arrow[r, "T_{n-1}(h)"] \arrow[d, "\phi_{n-1}"'] & T_{n-1}(K) \arrow[d, "\phi_{n-1}"] \\
(L_{n-1}F)(K') \arrow[r, "(L_{n-1}F)(h)"] & (L_{n-1}F)(K)
\end{tikzcd}$$

What we want to show is that the following diagram commutes:

$$\begin{tikzcd}
T_n(A') \arrow[r, "T_n(f)"] \arrow[d, "\phi_n"'] & T_n(A) \arrow[d, "\phi_n"'] \\
(L_nF)(A') \arrow[r, "(L_nF)(f)"] & (L_nF)(A)
\end{tikzcd}$$

But, this is the outer rectangle of the big diagram obtained by splicing together the three diagrams before, i.e.

$$\begin{tikzcd}
T_n(A') \arrow[rrr, "T_n(f)"] \arrow[dr, "\delta"'] \arrow[ddd, "\phi_n"'] & & & T_n(A) \arrow[dl, "\delta"] \arrow[ddd, "\phi_n"'] \\
& T_{n-1}(K') \arrow[r, "T_{n-1}(h)"] \arrow[d, "\phi_{n-1}"'] & T_{n-1}(K) \arrow[d, "\phi_{n-1}"] & \\
& (L_{n-1}F)(K') \arrow[r, "(L_{n-1}F)(h)"] & (L_{n-1}F)(K) & \\
(L_nF)(A') \arrow[ur, "\delta"] \arrow[rrr, "(L_nF)(f)"] & & & (L_nF)(A) \arrow[ul, "\delta"']
\end{tikzcd}$$

It is worth noting that all the small quadrilaterals in this large diagram commute: we mentioned, above, the commutativity of all but the leftmost and rightmost trapezoids – that commutativity follows from the way we defined the map ϕ_n.

If we now calculate $\delta\phi_n(A)T_n(f)$ and use all the commutativities, we get that equal to $\delta(L_nF)(f)\phi_n(A')$. But, we saw that δ is monic, so it can be cancelled from that equality and we are finished with the proof of the statement that ϕ_n is a natural transformation.

(**Aside:** It is worth remarking that the argument above, with $A = A'$ and $f = id$ shows that the map $\phi_n(A)$ doesn't really depend on the choice of the projective module P which maps onto A i.e. the "unnatural" definition is pretty natural after all!)

Finally, we need to show that the natural transformation commutes with the appropriate δ's so that the natural transformations can form a morphism of δ-functors.

So, suppose we are given a short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

we need to show that

$$\begin{array}{ccc}
T_n(A'') & \xrightarrow{\delta_n} & T_{n-1}(A'') \\
{\scriptstyle \phi_n(A'')}\downarrow & & \downarrow{\scriptstyle \phi_{n-1}}\ (A') \\
(L_nF)(A'') & \xrightarrow{\delta_n} & (L_{n-1}F)(A')
\end{array}$$

commutes.

Choose a short exact sequence

$$0 \to K'' \to P'' \to A'' \to 0$$

with P'' projective. Then we have:

$$\begin{array}{ccccccccc}
0 & \to & K' & \to & P'' & \to & A'' & \to & 0 \\
 & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow & & \|{\scriptstyle id} & & \\
0 & \to & A' & \to & A & \to & A'' & \to & 0
\end{array} \qquad (*\dagger)$$

where f arises because P'' is projective and g is induced by it and all the squares commute.

Using this, we get the following diagram:

$$\begin{array}{ccccc}
T_n(A'') & \xrightarrow{\delta_n} & T_{n-1}(K'') & \xrightarrow{T_{n-1}(g)} & T_{n-1}(A') \\
{\scriptstyle \phi_n(A'')}\downarrow & & {\scriptstyle \phi_{n-1}(K'')}\downarrow & & {\scriptstyle \phi_{n-1}(A')} \\
(L_nF)(A'') & \xrightarrow{\delta_n} & (L_{n-1}F)(K'') & \xrightarrow{(L_{n-1}F)(g)} & (L_{n-1}F)(A')
\end{array}$$

where the horizontal rows are not necessarily part of a complex. Notice that the righthand square commutes since ϕ_{n-1} is a natural transformation of functors, while the lefthand square commutes by the definition of the maps ϕ_n. We will be finished with the Theorem as soon as we establish the following

Claim: a) The composition in the top vertical row, $T_{n-1}(g)\delta_n$, is the connecting morphism δ_n for the sequence

$$0 \to A' \to A \to A'' \to 0$$

and the functors $\{T_n\}$.

b) The composition in the bottom vertical row, $(L_{n-1}F)(g)\delta_n$, is the connecting morphism δ_n for the sequence

$$0 \to A' \to A \to A'' \to 0$$

and the functors $\{L_nF\}$.

Pf: To see why this is true, let's use the fact that the functors $\{T_n\}$ are a homological δ- functor and apply that to the pair of short exact sequences $(*\dagger)$ above, to get

$$\begin{array}{ccccc} T''_n(A'') & \xrightarrow{\delta_n} & T_{n-1}(K'') & \to & T_{n-1}(P'') \\ {\scriptstyle id}\downarrow & & {\scriptstyle T_{n-1}(g)}\downarrow & & \\ T_n(A'') & \xrightarrow{\delta_n} & T_{n-1}(A') & \to & T_{n-1}(A) \end{array} .$$

The commutativity of the lefthand square in this diagram is precisely the first statement of the Claim.

The other part of the Claim is proved similarly.

With that Claim proved we have finished the proof of the main theorem. □