Ideal Generators of Monomial Curves in $\mathbb{P}^3$

Ping Li and Leslie Roberts

1. Outline

- What is a projective monomial curve in $\mathbb{P}^3$?
- Description of the ideal generators.
- distribution of quotients.
- Finiteness of segments.
- Application to the number of generators.

2. What is a projective monomial curve in $\mathbb{P}^3$?

$\mathscr{S} = \{a, b, d\}$ with $a, b, d \in \mathbb{N}$, $0 < a < b < d$ and $\gcd(a, b, d) = 1$.

$\boldsymbol{\alpha}_0 = (d, 0), \boldsymbol{\alpha}_1 = (d - a, a), \boldsymbol{\alpha}_2 = (d - b, b), \boldsymbol{\alpha}_3 = (0, d)$

$S \subset \mathbb{N}^2$ the semigroup generated by $\{\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2.\boldsymbol{\alpha}_3\}$.

$K[S] = K[s^d, s^{d-a}t^a, s^{d-b}t^b, t^d] \subseteq K[s, t]$.

$R = K[X_0, X_1, X_2, X_3]$

Define $\varphi : R \to K[s, t]$ by

$\varphi(X_0) = s^d, \varphi(X_1) = s^{d-a}t^a, \varphi(X_2) = s^{d-b}t^b, \varphi(X_3) = t^d$.

$\ker \varphi =: \mathfrak{p}$ is a homogeneous prime ideal in R, and $R/\mathfrak{p} = K[S]$.

$\mathcal{C} = \mathrm{Proj}(K[S]) \subset \mathbb{P}^3$.

Problem: How many generators does $\mathfrak{p}$ have?

$c = \gcd(a, b)$, $a' = a/c$, $b' = b/c$.

h be the smallest integer $\geq \lceil d/b' \rceil$ such that $hb' - d$ is divisible by a', $\ell = (hb' - d)/a'$.

$d = hb' - \ell a'$ with $h > 0, 0 \leq \ell < b'$.

3. Description of the generators of $\mathfrak{p}$.

Let $M = \begin{bmatrix} d & d-a & d-b & 0 \\ 0 & a & b & d \end{bmatrix} = (\boldsymbol{\alpha}_0|\boldsymbol{\alpha}_1|\boldsymbol{\alpha}_2|\boldsymbol{\alpha}_3)$

$M : \mathbb{Z}^4 \to \mathbb{Z}^2$.

$\mathscr{L}_{ij}$ sublattice of $\mathbb{Z}^2$ generated by $\boldsymbol{\alpha}_i$ and $\boldsymbol{\alpha}_j$, $i \neq j$

$\mathscr{L} = \mathscr{L}_{12} \cap \mathscr{L}_{03}$, $\mathscr{L}' = \mathscr{L}_{02} \cap \mathscr{L}_{13}$.

Clearly $\mathscr{L}_{03} = \{(x, y) \in \mathbb{Z}^2 | x \equiv 0, y \equiv 0 \mod d\}$ and if $(x, y) \in \mathscr{L}_{12}$, $x \equiv 0 \mod d$, then also $y \equiv 0 \mod d$. Thus $\mathscr{L} = \ker(\pi_2 : \mathscr{L}_{12} \to \mathbb{Z}/d\mathbb{Z})$ with π_2 onto so $(\mathscr{L}_{12} : \mathscr{L}) = d$.

Write $\langle x, y\rangle = x\boldsymbol{\alpha}_1 + y\boldsymbol{\alpha}_2, x, y \in \mathbb{Z}$. Then $\langle b', -a'\rangle, \langle -\ell, h\rangle \in \mathscr{L}$ and $\mathscr{L}$ is generated by $\{\langle b', -a'\rangle, \langle -\ell, h\rangle\}$.

If $\langle a_1, a_2\rangle \in \mathscr{L}, (a_1, a_2 \in \mathbb{Z})$ then there exist unique integers a_0, a_3 such that $a_1\boldsymbol{\alpha}_1 + a_2\boldsymbol{\alpha}_2 = a_0\boldsymbol{\alpha}_0 + a_3\boldsymbol{\alpha}_3$, equivalently $(-a_0 + a_1 + a_2 - a_3)^t \in \ker(M)$. From this one sees that π_{12} gives an isomorphism $\pi_{12} : \ker M \to \mathscr{L}$. Elements of $\ker(M)$ yield pure binomial elements of $\mathfrak{p}$ in an obvious way.

The ideal $\mathfrak{p}$ has a minimal set of pure binomial generators of two types.

(1) $X_1^{a_1}X_2^{a_2} - X_0^{a_0}X_3^{a_3}$, $a_1 \geq 0, a_2 \geq 0, a_1 + a_2 > 0, a_0 > 0, a_3 > 0$
(2) $X_0^{a_0}X_2^{a_2} - X_1^{a_1}X_3^{a_3}$, $a_i > 0, 0 \leq i \leq 3$

These generators are homogeneous in two senses. E.g. for the type one generator $X_1^{a_1}X_2^{a_2} - X_0^{a_0}X_3^{a_3}$ we have $a_1 + a_2 = a_0 + a_3$ and $a_1\boldsymbol{\alpha}_1 + a_2\boldsymbol{\alpha}_2 = a_0\boldsymbol{\alpha}_0 + a_3\boldsymbol{\alpha}_3$.

We will describe only the type one generators in terms of Hilbert bases. The type two can be described in a similar manner.

Let $\mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_3)$ denote the Hilbert basis (minimal generators) of the semigroup $\mathscr{L} \cap C(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_3)$ (in the $\boldsymbol{\alpha}_1$-$\boldsymbol{\alpha}_2$ coordinate plane). Then we have

Theorem: $K[S]$ is not Cohen-Macaulay if and only if $\mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_3)$ contains an element in the interior of the first quadrant (i.e. in $C(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$). Let A be the left most element of $\mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_3)$ in $C(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1)$ and B be the rightmost element of $\mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_3)$ in $C(\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3)$. Then in the non-Cohen-Macaulay case the type one generators are represented by A, B together with the elements of $\mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_3)$ in the interior of the first quadrant.

Some other facts are that $K[S]$ is not Cohen-Macaulay if and only if $\mathfrak{p}$ has four or more generators if and only if $\mathfrak{p}$ has a type two generator.

To see how this works we might look at a picture:

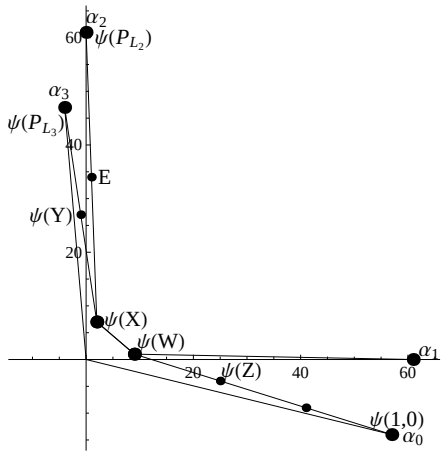


FIGURE 1. 12-basis diagram of $\mathscr{S} = \{14, 57, 61\}$

Here $\mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_3)$ contains seven elements, $\mathrm{A} = \langle 25, -4\rangle$, $\mathrm{B} = \langle -1, 27\rangle$ and there are two elements of $\mathscr{L}$ in the interior of the first quadrant, making 4 type one generators. Any of these points is easily turned into a "physical" generator. For example, $\langle -1, 27\rangle = -\boldsymbol{\alpha}_1 + 27\boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_0 + 25\boldsymbol{\alpha}_3$ so that $27\boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_1 - 25\boldsymbol{\alpha}_3 = 0$ and $X_2^{27} - X_0X_1X_3^{25} \in \mathfrak{p}$, which is a minimal binomial generator.

One can find $\mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_3)$ by an easy continued fraction calculation. Define a homomorphism $\psi : \mathbb{Z}^2 \to \mathscr{L}_{12}$ by $\psi(x_1, x_2) = x_1\langle b', -a'\rangle + x_2\langle -\ell, h\rangle = \langle x_1 b' - x_2\ell, -x_1 a' + x_2 h\rangle$. This is a monomorphism and has image $\mathscr{L}$. Furthermore $\psi(1,0) = \langle b', -a'\rangle = b'\boldsymbol{\alpha}_1 - a'\boldsymbol{\alpha}_2 = (b' - a')\boldsymbol{\alpha}_0$, and $\psi(c + \ell - h, b' - a') = \langle -(d - b), d - a\rangle = (b - a)\boldsymbol{\alpha}_3$. Let L_0 be the ray through $(1,0)$ (i.e. the positive horizontal axis) and L_3 be the ray through $(c + \ell - h, b' - a')$. Then $\psi(\mathrm{Hilb}_{\mathbb{Z}^2}(L_0, L_3)) = \mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_3)$. If L_3 is in the interior of the first quadrant (as is always the case if $K[S]$ is not Cohen-Macaulay) then $\mathrm{Hilb}_{\mathbb{Z}^2}(L_0, L_3)$ is the integer points on the boundary of the convex hull of the non-zero integer points in the first quadrant on or below L_3. The vertices of this convex hull correspond to the lower convergents of the continued fraction expansion of the slope of L_3 (together with $(1,0)$).

For the curve $\mathscr{S} = \{14, 57, 61\}$, we have $\ell = 16, h = 5$, and L_3 is the ray through $(12, 43)$. From the continued fraction $43/12 = \{3, 1, 1, 2, 2\}$ with convergents $\{3\} = 3, \{3, 1, 1\} = 7/2$, $\{3, 1, 1, 2, 2\} = 43/12$ so we find that the vertices of this convex hull are $(1,0), (1,3), (2,7), (12,43)$. The quotients correspond to integer points on the convex hull between the vertices. For example $q_4 = 2$ so there is one intermediate point between $(2,7)$ and $(12,43)$ namely $(7,25)$. Finally $\psi(1,0) = \langle 57, -14\rangle, \psi(3,1) = \langle 9, -1\rangle, \psi(2,7) = \langle 2, 7\rangle$ and $\psi(12,43) = \langle -4, 47\rangle$ yielding the vertices in the diagam (solid dots).

4. The distribution of quotients, following Knuth

Knuth Art of Computer Programming Volume 2 shows that a quotient in a continued fraction has value $\mathtt{a}$ with probability $\pi_{\mathtt{a}} = \log_2((\mathtt{a}+1)^2/((\mathtt{a}+1)^2-1))$. For $\mathtt{a} = 1, 2, 3, \ldots$ these values are $0.415, 0.170, 0.093\ldots$. Using this formula one checks

(1) $\Sigma\pi_{\mathtt{a}} = 1$.
(2) The average quotient should be $\Sigma\mathtt{a}\pi_{\mathtt{a}}$ but this sum diverges.
(3) Asymptotically the probability of a quotient lying in the interval $[10^{i-1}+1, 10^i]$ will decrease by a factor of 10 if i is increased by one.
(4) All continued fractions that occur in our discussion of the ideal generators of curves of degree d involve only integers less than d. Therefore we propose as a model for curves of degree d that quotient $\mathtt{a}$ occurs with probability $\log_2((\mathtt{a}+1)^2/((\mathtt{a}+1)^2-1))$, $1 \leq \mathtt{a} \leq d$, and 0 if $\mathtt{a} > d$.
(5) With this model the average quotient is now asymptotically $\log_2(d)$.

5. FINITENESS OF SEGMENTS

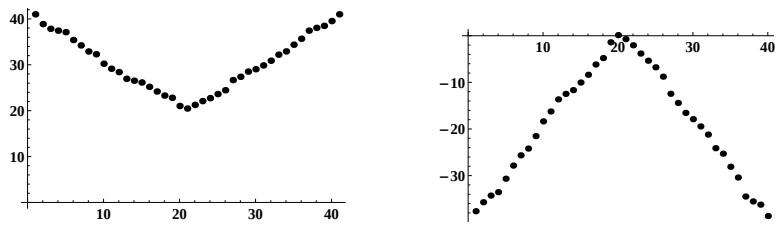
Number the vertices of $\mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_3)$ consecutively as $v_0,\ v_1, \cdots, v_i = (x_i, y_i), \cdots$ starting from the $\boldsymbol{\alpha}_0$ ray, and let $\alpha_i,\ i \geq 1$ be the angle (at the origin) between v_{i-1} and v_i. Then we can prove the following:

(1) As long as v_i and v_{i+1} are both in quadrants 4 or 1, $0 < x_{i+1} < x_i/(q_j + 1)$
(2) As long as v_i and v_{i+1} are both in quadrant 4, $0 < |y_{i+1}| < |y_i|/(q_j + 1)$, and as long as v_i and v_{i+1} are both in quadrant 1, $y_{i+1} > (q_k + 1)y_i$.

In the above, q_j and q_k are some quotient of the continued fraction of the slope of L_3 (and hence a positive integer). From this we see that $|v_i|$ decreases exponentially in quadrant 4 and (by symmetry) increases exponentially in quadrant 2. Within the first quadrant x_i decreases exponentially and y_i increases exponentially, with the larger dominating. Thus we expect an abrupt change from exponential decrease of $|v_i|$ to exponential increase, the nature of which is independent of d.

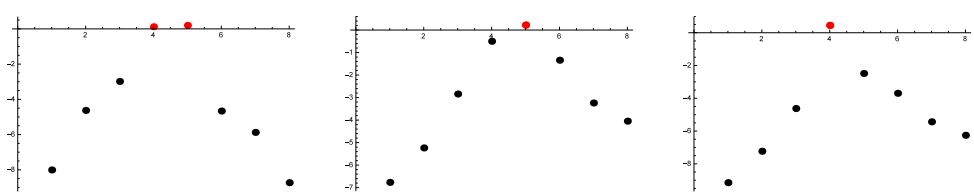
We have $q_j d = |v_{i-1}||v_i| \sin(\alpha_i)$, from which it follows that the α_i grow exponentially and then abruptly switch to exponential decrease, at about twice the rate of the change in $|v_i|$. From this formula it is obvious that $|v_i|$ cannot be less than $\sqrt{d}$ for two consecutive values of i. One can also show that on average the minimum value of $|v_i|$ has to decrease almost to $\sqrt{d}$ (otherwise the sum of the angles will not be large enough to get through the first quadrant).

To see how this works in practice we plot graphs of $\log_{10}$ of the norms of the vertices and $\log_{10}$ of the angles of $\mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_3)$ for a random curve of degree $d = 10^{41} + 37$.



The norms start at about 10^{41}, decrease roughly exponentially to about $10^{20} \approx \sqrt{10^{41}}$, and switch abruptly to exponential growth. Similarly the angles start at about $1/d = 10^{-41}$, increase exponentially to just over one radian, then abruptly switch to exponential decrease. All "randomly chosen" curves that we have examined behave in a similar manner.

More crucial is how the angles behave as we go through the first quadrant. Below are graphs of the logs of the angles, for three "randomly chosen" curves of degrees 10^{200}, 10^{1600}, 10^{10003} respectively. The red dots are angles whose subtending vertices are in different quadrants, and the black are angles with subtending vertices in the same quadrant.



As expected there is no obvious scaling with d.

The element $\langle b', -a' \rangle$ of $\mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_3)$ on the $\boldsymbol{\alpha}_0$ ray is pretty much randomly distributed over the half fourth quadrant above $y = -x$. Given the exponential growth of the α_i, the last angle in quadrant 4 will usually be close to the initial starting angle $\tan^{-1}(a/b)$. The situation is similar if we start on the $\boldsymbol{\alpha}_3$ ray, so averaged over all curves of degree d we expect a very small number of vertices of $\mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_3)$ in the second quadrant. This is illustrated by the graphs above, and some experimental data. In a sample of about 60000 randomly chosen curves of degrees varying from 10^{10} to 10^{40} we obtained that $0.62, 0.34, 0.03, 0.003, 0.00015$ of the curves have $m = 0, 1, 2, 3, 4$ vertices respectively in the interior of $C(L_1, L_2)$. (The Cohen-Macaulay curves contribute about half of the $m = 0$ cases.) No curve in our samples had $m > 4$. For the same samples the segments contributing type one generators are distributed as in the following table

Table 1

	-1	0	1	2	3	4
Average	0.3064	0.3203	0.3331	0.0376	0.0026	0.0001

In this table -1 means Cohen-Macaulay, i.e. no elements of $\mathscr{L}$ in the interior of the first quadrant. This means that one segment of $\mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_3)$ crosses the first quadrant and has no subdivision point in the interior. If there is only one element of $\mathscr{L}$ in the interior of the first quadrant this is counted as 0 segments. Either one segment of $\mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_3)$ crosses the first quadrant and has only one subdivision point in the interior, or there is one vertex in the interior of the first quadrant and the two subtending segments have no other vertex in the interior.

One might wonder how well Knuth's quotient distribution fits the ideal generator quotients. This is addressed in the following tables. The K row is Knuth's distirbution, and the Avg row is the 60000 curves sample mentioned above. This means for example that Knuth predicts that .8745 of the quotients have value from 1 to 10, and that for that sample .8737 of the quotients have value from 1 to 10. The 10^6 row is from a sample of 10^6 curves of degree 10^6.

Table 2

	1 to 10	11 to 100	101 to 1000	10^3+1 to 10^4	10^4+1 to 10^5
K	0.8745	0.1113	0.01277	1.296×10^{-3}	1.298×10^{-4}
$d=10^6$	0.8759	0.1102	0.01256	1.232×10^{-3}	1.045×10^{-4}
Avg	0.8737	0.1116	0.0132	1.296×10^{-3}	$.8\times10^{-4}$

Table 3

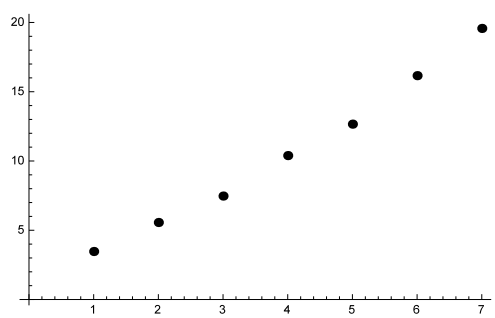
	1	2	3	4	5	6	7
K	0.4150	0.1699	0.0931	0.0589	0.0406	0.0297	0.0227
Avg	0.4120	0.1714	0.0915	0.0607	0.0414	0.0306	0.0229

Since $m \geq 2$ only about .03 of the time most of the quotients tabulated in Tables 2 and 3 come from parts of segments of $\mathrm{arc}_{\mathscr{L}}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_3)$. These quotients follow Knuth's distribution closely. However the distribution of quotients from segments of $\mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_3)$ that lie entirely in the first quadrant is quite different from Knuth, for example, about 0.72 of such quotients in our six large samples have value 1.

6. Application to generators

We have observed an average of about .3 type one generator segments per curve, and type two generators will double this. Knuth's quotient distribution is observed to approximately hold for such segments, so we expect the average number of ideal generators of all curves of degree d to behave somewhat like Knuth's distribution when there are many generators. For example, we expect that the fraction of all curves with number of generators in the interval $[10^i, 10^{i+1}]$ to be about .1 of the fraction with number of generators in the interval $[10^{i-1}, 10^i]$, so that each bin will contribute about the same amount to the average number of generators (up to d) and that the average number of generators should grow proportional to $\log_{10} d$ (the number of bins). However we need very large sample sizes to have a chance of having representatives in the larger bins, and hence to obtain a reasonable average.

We illustrate this by plotting the average number of generators as a function of $i = \log_{10} d$ for all curves of degrees 10 and 100 and samples with 10^i curves of degree 10^i, $3 \leq i \leq 7$. The graph is approximately a straight line, as expected.



For our sample of 60000 curves the fraction of all curves with i generators $3 \leq i \leq 12$ is given by the following table. (None of the curves had 2 generators). There was no obvious dependence on d.

Table 4

d	3	4	5	6	7	8	9	10	11	12
AVG	0.3064	0.1171	0.1050	0.0874	0.0690	0.0541	0.0404	0.0316	0.0240	0.0193

We do not see how to predict the above values. When the data is binned the distribution behaves somewhat as expected. For the 10^7 curves of degree 10^7 even the largest possible bin is well represented, although with some shortfall. For the 60000 curves the largest two bins in the table are not represented at all, presumably because the sample size is not large enough. After all, that .0001 comes from only 6 curves. We would need a sample size of about 10^{40} to get reasonable values for the largest bins.

Table 5

d	2 to 10	11 to 100	101 to 1000	1001 to 10^4	10^4+1 to 10^5	10^5+1 to 10^6	10^6+1 to 10^7
10^7	0.8134	0.1742	0.0111	1.061×10^{-3}	1.029×10^{-4}	0.86×10^{-5}	0.8×10^{-6}
AVG	0.8109	0.1761	0.0117	0.0012	0.0001	0	0

References

(1) P. Li, D.P. Patil, and L. Roberts. Bases and ideal generators for projective monomial curves. Communications in Algebra, 40(1):173(191), 2012.

(2) I.Peeva and B.Sturmfels. Syzygies of codimension 2 lattice ideals. Mathematische Zeitschrift, 229:163(194), 1998.

In (1) the ideal generators of $\mathfrak{p}$ were found in a slightly different from than in our Theorem. In (2) the ideal generators were also described in terms of Hilbert bases. In our language their type one generators come out mostly as $\mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_2) \cap \mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_3)$ whereas we describe them as $\mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_3) \cap \mathrm{Hilb}_{\mathscr{L}}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$ (non-Cohen-Macaulay case only).