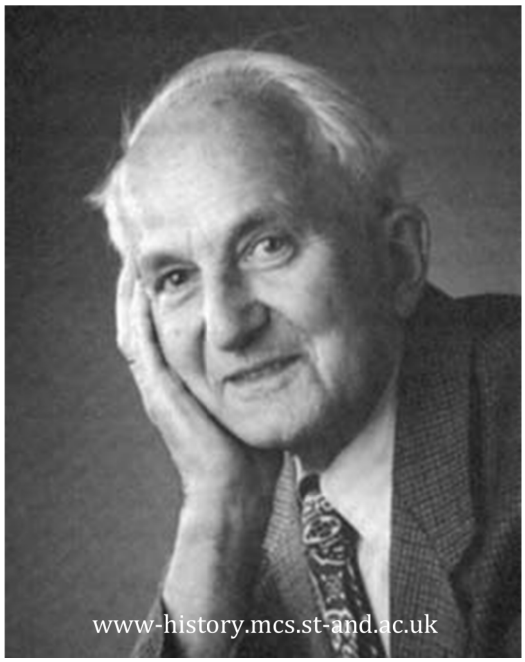
Collatz Generalized



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Introduction

If an integer is odd, multiply by 3 and add 1 to get a new integer. If it is even, divide by two. It is conjectured that no matter what positive integer you begin with, you will eventually reach 1. This problem, despite being so easily understood, turns out to be difficult, if not impossible, to prove or disprove. For over half a century, it has enticed mathematicians of every caliber, from grade school children to well-published Ph.D. degree holders. Among them is the author of this undergraduate research paper. This report considers the conjecture in the context of a generalization: when an integer is odd, we multiply by A and add B. In this paper, we investigate whether the same goal - that every positive integer iterates to a given number - can be achieved for values of A and B other than 3 and 1, respectively. In turn, this generalization gives a better perspective on why the original conjecture may or may not be true.

The Collatz Problem

Also known as the 3x+1 problem, the Syracuse problem, Kakutani's problem, Hasse's algorithm, and Ulam's problem, the Collatz problem is an unsolved mathematical problem generally credited to Lothar Collatz. It is based on the following function, which we will refer to as the "3x+1 function":

$$C(x) = \begin{cases} 3x+1, & if\ x \equiv 1 \text{ (mod 2)}, \\ \dfrac{x}{2}, & if\ x \equiv 0 \text{ (mod 2)}. \end{cases}$$

The Collatz conjecture states that every positive integer repeatedly mapped under $C(x)$ eventually iterates to 1. Although the conjecture has been verified for every positive integer up to at least 10^{18} [11], it remains to be proven as true for all positive integers.

There are various methods of approaching the conjecture, many of which utilize trajectories. The *trajectory* (or *orbit*) of integer x, denoted $\langle x \rangle$, is the ordered sequence $\{x_i\}$ defined by $x_1 = x$ and $x_{i+1} = C(x_i)$ for $i \in \mathbb{Z}^+$. This definition brings us to the first of two variations of the Collatz conjecture that will be the focus of this project:

Collatz Conjecture –Trajectory Variation: <u>For all $x \in \mathbb{Z}^+$, $1 \in \langle x \rangle$.</u>

For later use, we will also need a few definitions regarding trajectories. First of all, a trajectory is *divergent* if there are no integers that appear twice in the trajectory. Otherwise, the trajectory is *convergent* and the repeating ordered set of numbers that it contains is called a *loop*. For example, $\langle 3 \rangle = \{3, 10, 5, 16, 8, 4, 2, 1, 4, \ldots\}$ converges and contains the loop $\{4, 2, 1\} = \{2, 1, 4\} = \{1, 4, 2\}$. Let us note that these definitions of convergence and divergence are not to be confused with the substantially different definitions generally found in mathematics.

Methods of approaching the Collatz problem also frequently utilize the following alternate forms of the 3x+1 function:

$$D(x) = \begin{cases} \dfrac{3x+1}{2}, & if\ x \equiv 1 \pmod 2, \\ \dfrac{x}{2}, & if\ x \equiv 0 \pmod 2; \end{cases}$$

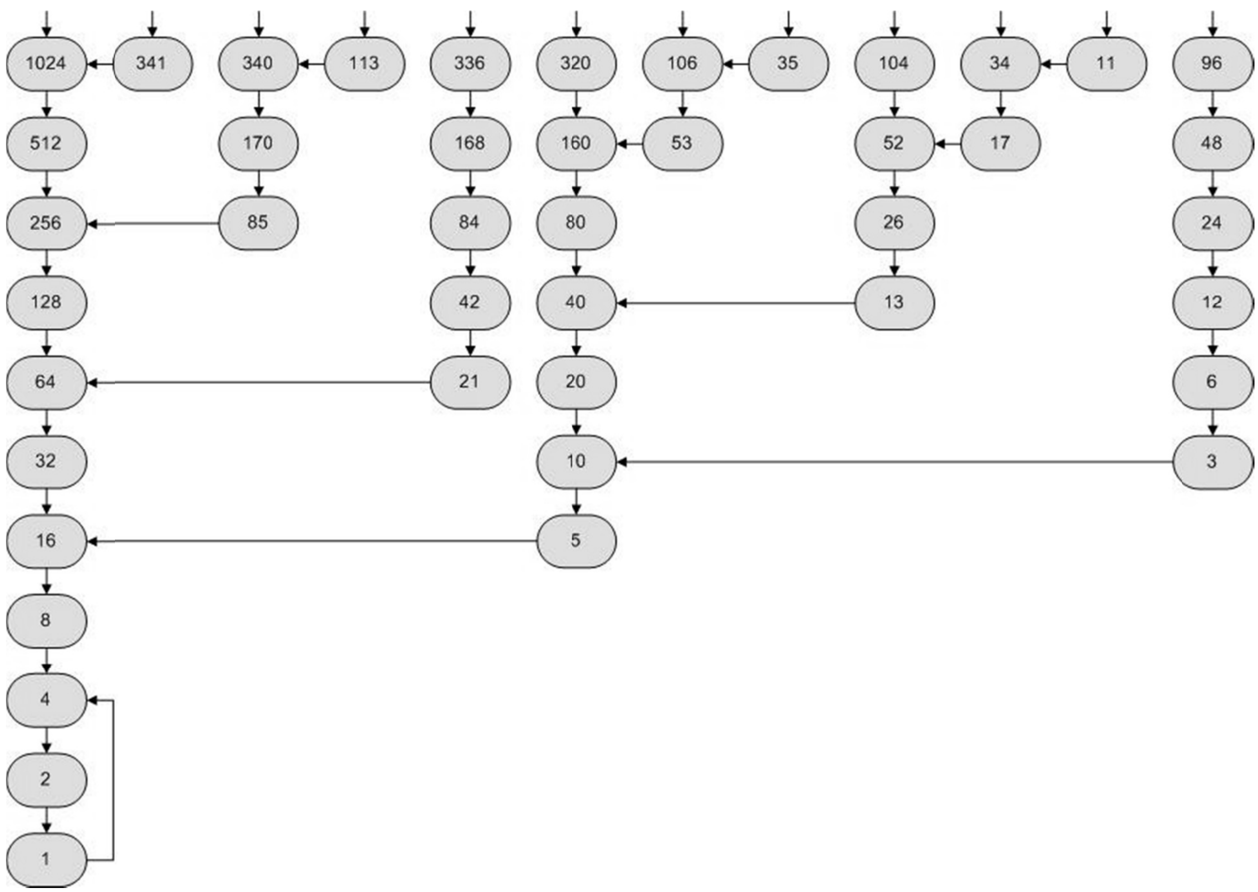
$$T(x) = \frac{3x+1}{2^{m(3x+1)}}, \quad if\ x \equiv 1 \pmod 2,$$

where $m(x)=\max(k \in \mathbb{Z}^+ : 2^k | x)$. When x is odd, $C(x)$ is even, so $D(x)$ results in more rapid convergence by dividing $3x$+1 by 2 within a single iteration. $T(x)$ takes the idea even further, dividing by the highest possible power of 2 and thus eliminating even integers from the problem altogether. These function varieties are interchangeable in practice, because they do not alter which odds appear in a trajectory or the order in which the odd elements of a trajectory appear. Thus, a trajectory's convergence or divergence and whether a trajectory contains 1 do not depend on which of the three forms is used.

Collatz Tree

An alternate way to approach the 3x+1 problem is with the Collatz tree (Figure 1), which is generated via inverse calculation of $C(x)$, beginning with 1.

Figure 1 – Collatz Tree



The Collatz tree provides a visual representation of trajectories, and leads us to our second variation of the Collatz conjecture:

> *Collatz Conjecture – Tree Variation*: <u>The Collatz tree is connected.</u>

By *connected*, we mean that for all $x, y \in \mathbb{Z}^+$, $C^n(x) = C^m(y)$ for some $n, m \in \mathbb{Z}^+$.

Project Summary

Rather than investigating the trajectories of positive integers under the 3x+1 function, this paper explores trajectories under a generalized "Ax+B function" on the integers, defined by

$$C_{A,B}(x) = \begin{cases} Ax + B, & if\ x \equiv 1 \ (\text{mod } 2), \\ \dfrac{x}{2}, & if\ x \equiv 0 \ (\text{mod } 2), \end{cases}$$

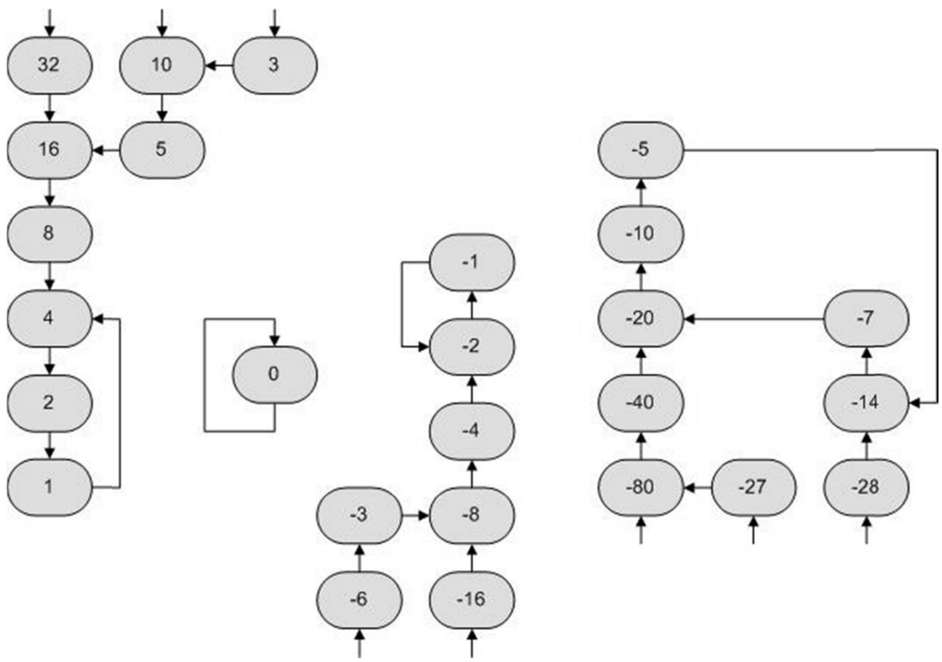
where $A, B \in \mathbb{Z}$. The purpose of this generalization is to create context for a better understanding of the 3x+1 function – the case where $A = 3$ and $B = 1$. Other mathematicians have explored various cases of the Ax+B function, including 3x+B by Belaga and Mignotte [1], 5x+1 by Conrow [3] and by Volkov[10], and Ax+1 by Crandell [6] and by Franco and Pomerance [7]. We will incorporate some of their work into this project.

To begin with, we will show that it is reasonable to only consider positive integers, and to only use odd value for A and B. Then we will break the generalization into three broad categories: $A < 3$, $A = 3$, and $A > 3$. After gaining an understanding of the convergence or divergence trends in each category, we will finish by evaluating the generalization in the context of the Collatz conjecture – both the trajectory variation and the tree variation.

Initial Parameter Restriction

Consider the 3x+1 function when applied to all integers, instead of just positive integers. The Collatz conjecture states that every trajectory contains 1. But if we allow for non-positive x, then there exist at least three loops – and thus infinite trajectories – which do not contain 1: {0}, {-1, -2}, and {-5, -14, -7, -20, -10} (Figure 2).

Figure 2 – Collatz Tree for All Integers



Thus considering $x < 1$ in the generalization does not contribute to understanding of the unsolved problem. Furthermore, if the domain of the Ax+B function is restricted to positive integers, then the range should be similarly restricted, else numerous trajectories would terminate as finite sequences. To ensure that $C_{A,B}(x) > 0$ for all positive integers x, both A and B must be non-negative, and cannot both be 0. If $A = 0$, then every trajectory immediately converges with loop $\{B\}$, which is neither an interesting nor a useful result. Thus $A > 0$ and $B \geq 0$.

Next we eliminate the need to consider the Ax=B function when A and B are not both odd. Let either A or B but not both be even, and let x be odd. Then $Ax + B$ is odd. Also, $Ax + B \geq x$ since $A \geq 1$, $x \geq 1$ and $B \geq 0$. If $A = 1$ and $B = 0$, then $\langle x \rangle = \{x, x, x, \ldots\}$ which is an unproductive result. Otherwise, $Ax + B$ is strictly greater than x, so the trajectory of x consists of increasing odd values, and will therefore invariably diverge. Now let x be even. Since $x > 0$, $C_{A,B}(x) < x$, and so the trajectory will contain decreasing values until it reaches an odd value, at which point, it will diverge as previously demonstrated. Even if x is a power of 2 and will iterate to 1 before any other odd value, it will still diverge upon reaching 1 (a conundrum we consider later). Thus every trajectory diverges under the Ax+B function when exactly one of A and B is even, so A and B must be either both even or both odd. Given that restriction, $C_{A,B}(x)$ is even for odd x, so we can utilize the D and T functions with the generalization:

$$D_{A,B}(x) = \begin{cases} \dfrac{Ax+B}{2}, & if\ x \equiv 1 \pmod 2, \\ \dfrac{x}{2}, & if\ x \equiv 0 \pmod 2; \end{cases}$$

$$T_{A,B}(x) = \frac{Ax+B}{2^{m(Ax+B)}}, \quad if\ x \equiv 1 \pmod 2,$$

where $m(x) = \max(k \in \mathbb{Z}^+ : 2^k | x)$. Returning to the parameters, suppose both A and B are even and let x be odd. Then

$$D_{A,B}(x) = \frac{Ax+B}{2} = \frac{A}{2}x + \frac{B}{2} = C_{\frac{A}{2},\frac{B}{2}}(x).$$

Thus the trajectories are equivalent, with regards to odd elements, under $C_{A,B}(x)$ and $C_{\frac{A}{2},\frac{B}{2}}(x)$, and so we restrict our investigation to $A, B, x \in \mathbb{Z}^+$, where A and B are both odd.

With these restrictions on the parameters, we begin investigating convergence and divergence for various values of A and B. In particular, we will explore the general situation when $A < 3$, when $A = 3$, and when $A > 3$.

$\boldsymbol{A < 3}$

Since A is positive and odd, this category only deals with $A = 1$. In this situation we can prove that every trajectory converges and that there are finitely many x that can be in a loop for any given B.

Theorem 1:

Let $A = 1$. Then for all odd $B \in \mathbb{Z}^+$,

a. $\langle x \rangle$ converges for all $x \in \mathbb{Z}^+$, and
b. under $D_{1,B}(x)$, $x \in \mathbb{Z}^+$ is in a loop iff $x \leq B$.

Proof:

Let $x > B$. If x is even, then $\frac{x}{2} < x$. If x is odd, then $\frac{x+B}{2} < \frac{x+x}{2} = x$. Thus $D_{1,B}(x) < x$. Therefore, every trajectory converges (part a) and x is in a loop only if $x \leq B$ (one direction of part b).

Let $x \leq B$. We want to show that x is in a loop. Since there are finitely many $x \leq B$, it suffices to show for each $x \leq B$ that $D_{1,B}(x) \leq B$ and that there exists a positive integer $y \leq B$ such that $D_{1,B}(y) = x$. If x is even, then $D_{1,B}(x) = \frac{x}{2} < x \leq B$. If x is odd, then $D_{1,B}(x) = \frac{x+B}{2} \leq \frac{B+B}{2} = B$. That satisfies the first requirement, and so we continue to the second. If $x < \frac{B}{2}$, then $D_{1,B}(2x) = x$ and $0 < 2x \leq B$. If $x > \frac{B}{2}$, then $D_{1,B}(2x - B) = x$ and $0 < 2x - B \leq B$. That satisfies the second requirement. Thus x is in a loop if $x \leq B$ (the other direction of part b). ■

$\boldsymbol{A = 3}$

As with the last section, we are only dealing with one value of A. However, unlike the last section, our claim might not be true, and even if it is, it might not be provable.

Conjecture 1:

Let $A = 3$. Then for all odd $B \in \mathbb{Z}^+$, $\langle x \rangle$ converges for all $x \in \mathbb{Z}^+$.

Let the *stopping time* of $x \in \mathbb{Z}^+$ under $D_{A,B}(x)$, denoted $\chi_{A,B}(x)$, be the least number of iteration required to reach a number less than x. That is,

$$\chi_{A,B}(x) = \begin{cases} \min\{k \in \mathbb{Z}^+ : D_{A,B}^k(x) < x\}, & if\ \{k \in \mathbb{Z}^+ : D_{A,B}^k(x) < x\} \neq \emptyset, \\ \infty, & if\ \{k \in \mathbb{Z}^+ : D_{A,B}^k(x) < x\} = \emptyset. \end{cases}$$

Thus a stronger variation of Conjecture 1 would be: Given $A = 3$ and any odd $B \in \mathbb{Z}^+$, there exists an $N \in \mathbb{Z}^+$ such that $\chi(x)$ is finite for all $x > N$. If every integer greater than N iterates to some smaller integer, then not only does every trajectory converge, but only a finite number of loops exist. We cannot prove this, but we can provide strong evidence for convergence through a slightly weaker statement. Consider the following sequence:

$$F_B(k) = \lim_{m \to \infty} \frac{|\{x \leq m : \chi_{3,B}(x) \geq k\}|}{m}.$$

For each k, $F_B(k)$ is the density of positive numbers with stopping time not less than k. Terras [13,14] proved that $\lim_{k\to\infty} F_1(k) = 0$, which means the density of positive integers with finite stopping time is 1. Furthermore, Terras' proof can be expanded for all odd positive B, which we will we state here as a theorem, without reconstructing the extensive proof.

Theorem 2:

For all odd $B \in \mathbb{Z}^+$, $\lim_{k\to\infty} F_B(k) = 0$.

$\boldsymbol{A > 3}$

Now we deal with all other values of A. When $A > 3$, it appears that there exist divergent trajectories. For example, under $T_{5,1}$, it seems $\langle 7 \rangle = \{7,9,23,29,73,183, \dots\}$ diverges. In Figure 3, we see that after 300 iterations, $\langle 7 \rangle$ has elements greater than 10^{25}, and is reaching exponentially larger values at a fairly steady rate. This is also the case for numerous other trajectories when $A > 3$, which leads us to our next conjecture.

Conjecture 2:

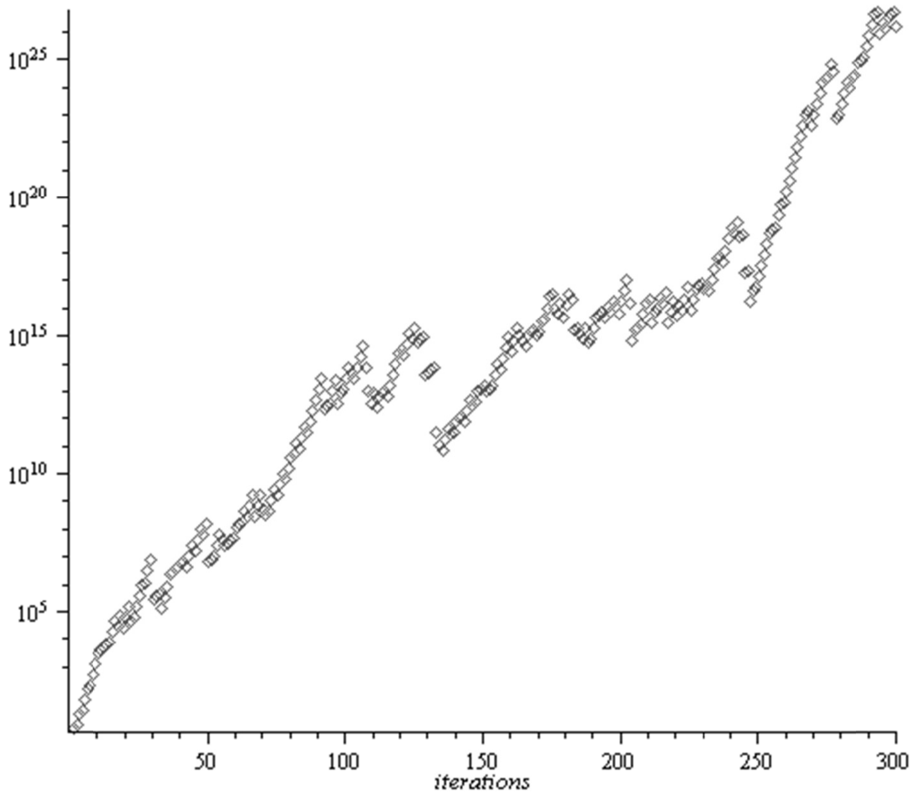
Let $A > 3$. Then for any odd $A, B \in \mathbb{Z}^+$,

a) (Weak) there exist $x \in \mathbb{Z}^+$ for which $\langle x \rangle$ diverges; or
b) (Strong) there exists a positive density of x in $\mathbb{Z}^+$ for which $\langle x \rangle$ diverges.

Comment:

We have two different strengths of the conjecture to illuminate the fact that, although the stronger version appears to be true, the substantially weaker version remains unproven.

Figure 3 – Apparent Divergence of $\langle 7 \rangle$ under $T_{5,1}$



A Heuristic Argument:

Note that for every other odd x, $2 \mid (Ax + B)$, but $4 \nmid (Ax + B)$. Likewise, for every fourth odd x, $4 \mid (Ax + B)$, but $8 \nmid (Ax + B)$. Treating a trajectory as a random walk, we follow this pattern and establish that $P\left(T_{A,B}(x) = 2^{-k}(Ax + B)\right) = 2^{-k}$. Thus for high enough values of x,

$$E\left(\frac{T_{A,B}(x)}{x}\right) = \sum_{k=1}^{\infty} \frac{2^{-k}(Ax + B)}{x} 2^{-k} = \sum_{k=1}^{\infty} \frac{Ax + B}{4^k x} > \sum_{k=1}^{\infty} \frac{A}{4^k} = \frac{\frac{A}{4}}{1 - \frac{1}{4}} = \frac{A}{3},$$

since $\sum_{k=1}^{\infty}\left(\frac{A}{4^k}\right)$ is a geometric series. This implies that every trajectory will tend to diverge for all $A > 3$.

Yet regardless of the “rate of divergence” suggested by our argument, there is no guarantee that any given trajectory will never reach the product of a loop element and a power of 2.

Goal I: Every trajectory contains 1

With a general understanding of convergence and divergence when $A < 3$, $A = 3$, and $A > 3$, we now move on to identifying what combinations of A and B satisfy the Collatz conjecture. For this section, we will first consider the trajectory variation:

<u>For all $x \in \mathbb{Z}^+$, $1 \in \langle x \rangle$.</u>

Theorem 3:

For all $x \in \mathbb{Z}^+$, $1 \in \langle x \rangle$ only if $B = 1$.

Proof:

Let $x \in \mathbb{Z}^+$ and let $B|x$. Note that $B|(Ax + B)$. Also, if x is even, $\left| \frac{x}{2} \right.$, since B is odd. Thus $B|C_{A,B}(x)$. Therefore, $B|C_{A,B}^n(x)$ for all $n \in \mathbb{Z}^+$. From that we know that B divides every element of $\langle B \rangle$. Therefore, 1 is not in $\langle B \rangle$ unless $B = 1$. ■

By Theorem 3, we assume $B = 1$ if we want 1 in every trajectory. Then by Theorem 1, every trajectory converges when $A = 1$, and only $x \leq B = 1$ are in a loop. Thus, the trajectory variation of the Collatz conjecture is true for $(A, B) = (1,1)$. Now $(A, B) = (3,1)$ is the original unsolved problem, so we move on to the $(A, 1)$ scenarios for $A > 3$. If we could prove Conjecture 2b, we would be done. Lacking that, however, we turn to other methods.

One way to show that trajectories exist which do not contain 1 is to show the existence of loops which do not contain 1. For $B = 1$, the only such known loops are $\{13,33,83\}$ and $\{17,43,27\}$ under $T_{5,1}(x)$ and $\{27,611\}$ and $\{35,99\}$ under $T_{181,1}(x)$.

Franco and Pomerance [7] detail a second method of finding values of A for which there exist trajectories without 1. Consider the case where 1 is the second iteration of x under $T_{A,1}(x)$. Then for some $k, l \in \mathbb{Z}^+$,

$$\frac{A\left(\frac{Ax+1}{2^k}\right)+1}{2^l} = 1$$

$$\implies A(Ax+1) = 2^k(2^l - 1).$$

Then since A is odd, $A|(2^l - 1)$. From number theory, we know that $A|(2^l - 1)$ only if $\varphi(A)|l$, where φ is the Euler totient function. Define d as

$$d = \gcd\left(\frac{2^{\varphi(A)} - 1}{A}, A\right).$$

Suppose A is such that $d \neq 1$. Then for all $l \in \mathbb{Z}^+$ for which $A|(2^l - 1)$,

$$d \,\middle|\, \frac{2^l - 1}{A}.$$

But since A is odd, d is odd, so $\gcd(2^k, d) = 1$. Thus $d|(Ax + 1)$, which is a contradiction, since $d|Ax$. Therefore, if $d \neq 1$ for some A, then 1 is not the second iteration of any x under $T_{A,1}(x)$. Franco and Pomerance refer to such A as Wieferich numbers. Some sources [16] define A as a Wierferich number only when $d = A$. However, for the purposes at hand, we will stick with the first definition. Here are the first fifty Wieferich numbers:

21, 39, 55, 57, 105, 111, 147, 155, 165, 171, 183, 195, 201, 203, 205, 219, 231, 237, 253, 273, 285, 291, 301, 305, 309, 327, 333, 355, 357, 385, 399, 417, 429, 453, 465, 483, 489, 495, 497, 505, 507, 525, 543, 555, 579, 597, 605, 609, 615, 627.

All in all, we know that the trajectory variation of the Collatz conjecture is false if $B \neq 1$. When $B = 1$, the conjecture is true for $A = 1$ and appears to be true for $A = 3$. Furthermore, the conjecture appears to be false for all $A > 3$, though we have only proven that to be the case for $A = 5$, $A = 181$, and when A is a Wieferich number.

Goal II: The Collatz tree contains every positive integer

Aside from the trivial case of $A = B = 1$, the trajectory variation of the Collatz conjecture (Goal I) appears to be uniquely satisfied by the 3x+1 function. However, recall from our initial discussion of the parameters that it is possible for 1 to be in a divergent trajectory. This fact alone gives us reason to rethink the goal of our generalization, which is why we have another variation of the Collatz conjecture – the tree variation:

The Collatz tree is connected.

Even if 1 is not in a loop, it may be possible for a Collatz tree to be connected. From the discussion of Goal I, we know that the tree is connected when $(A, B) = (1,1)$, since everything converges to 1. However, if $A = 1$ and $B > 1$, then $D_{1,B}(B) = B$ and $D_{1,B}(x) < B$ for $x < B$, which implies at least 2 disjoint "branches". For example, look at the trees for $C_{1,3}(x)$ and $C_{1,5}(x)$ (Figure 4).

Figure 4a – Collatz Tree of $C_{1,3}(x)$

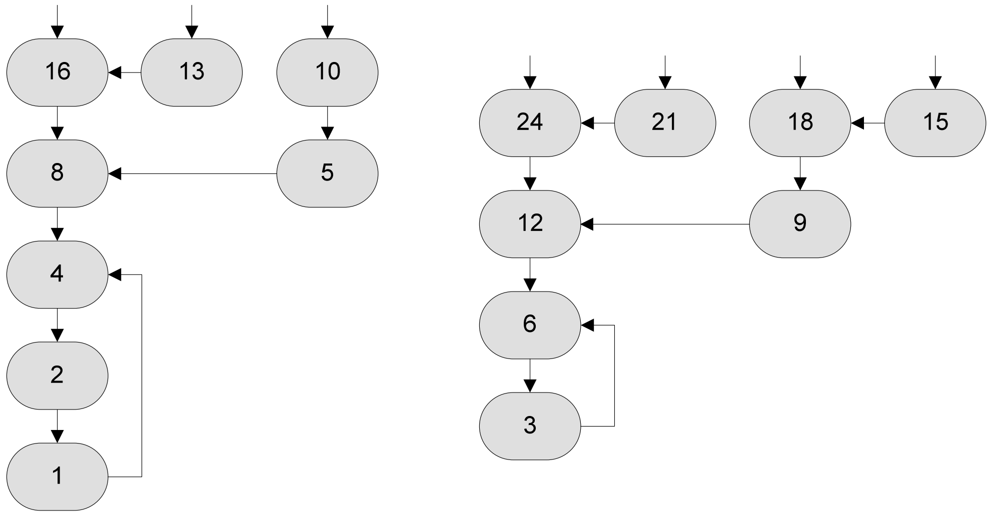
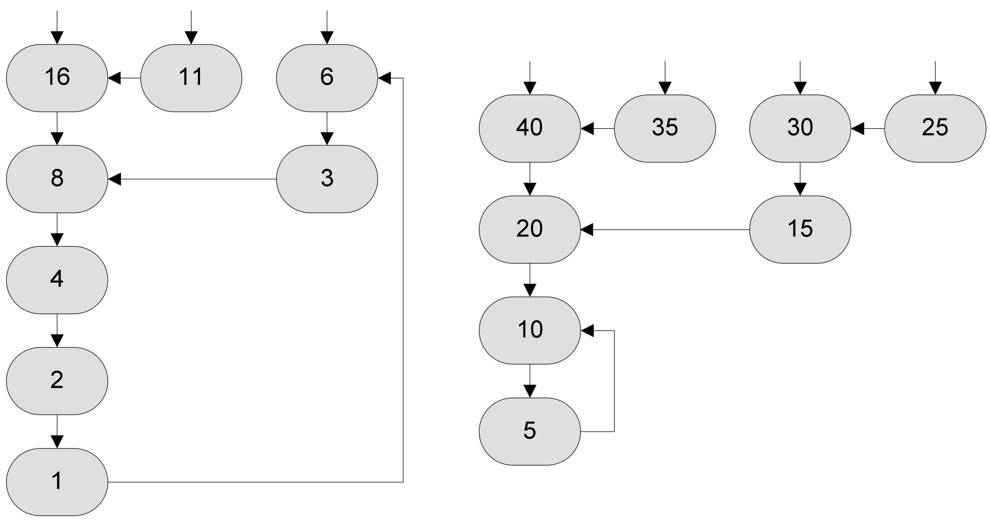
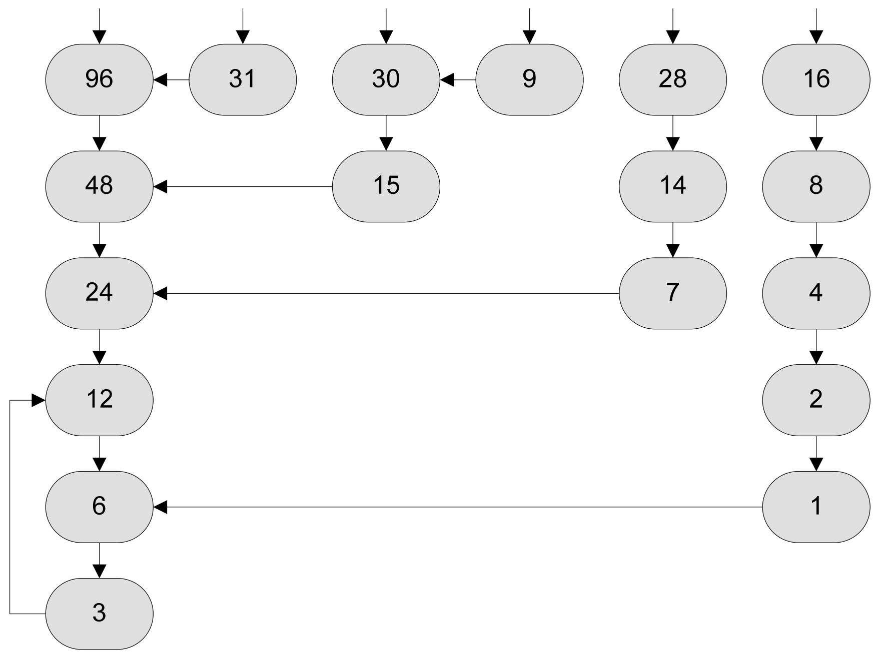


Figure 4b – Collatz Tree of $C_{1,5}(x)$



So for $A = 1$, the trajectory and tree variations of the Collatz conjecture are satisfied by the same B, namely $B = 1$. This may not be the case with $A = 3$. Consider the tree of $C_{3,3}(x)$ (Figure 5). Although 1 is not in a loop, and thus there are many trajectories without 1, it appears that every trajectory converges to the loop containing 3, in which case, the Collatz tree for $(A, B) = (3,3)$ would be connected. The author has verified with Maple that $\langle x \rangle$ converges to the loop with 3 for all $x < 10^8$ (see the Appendix).

Figure 5 – Collatz Tree of $C_{3,3}(x)$



However, if we find that more than one loop exist when $B = B'$, than using Theorem 4, we know that more than one loop exists whenever B is a multiple of B'. For example, the existence of multiple loops under $T_{3,5}$ ({1}, {5}) and $T_{3,7}$ ({5,11}, {7}) implies that multiple loops exists under $T_{3,15}$, $T_{3,21}$, $T_{3,25}$, $T_{3,35}$, $T_{3,45}$, $T_{3,49}$, etc.

Theorem 4:

Let $\{x_1, x_2, \dots, x_n\}$ be a loop under $T_{A,B}$. Then for all $p \in \mathbb{Z}^+$, $\{px_1, px_2, \dots, px_n\}$ is in a loop under $T_{A,pB}$.

Proof:

Note that $x_1 = T_{A,B}(x_n) = \frac{Ax_n+B}{2^{k_1}}$ for some $k_1 \in \mathbb{Z}^+$. Likewise for each $i \in \{2,3,\dots,n\}$, $x_i = T_{A,B}(x_{i-1}) = \frac{Ax_{i-1}+B}{2^{k_i}}$ for some $k_i \in \mathbb{Z}^+$. Thus

$$px_1 = p\frac{Ax_n + B}{2^{k_1}} = \frac{Apx_n + pB}{2^{k_1}} = T_{A,pB}(px_n);$$

$$px_i = p\frac{Ax_{i-1} + B}{2^{k_i}} = \frac{Apx_{i-1} + pB}{2^{k_i}} = T_{A,pB}(px_{i-1})$$

for all $i \in \{2,3,\dots,n\}$. ■

Nevertheless, eliminating the requirement that all trajectories iterating to 1 allows for the possibility of more values of B that are of interest when $A = 3$. For $A > 3$, we return to the fact that there appear to be divergent trajectories, which implies a disconnected Collatz tree. However, having eliminated the necessity for $B = 1$, and with the method involving Wieferich numbers no longer at our disposal, a connected tree is now more plausible for $A > 3$.

Furthermore, we might reconsider the possibility of incorporating non-positive integers into the generalization. Is it possible to find a combination of A and B, such that a Collatz tree of ever integer, not just positive integers, is connected? Since $C_{A,B}(0) = 0$ for all (A,B), then a connected tree would require every trajectory to contain 0. Obviously, the trivial case $(A,B) = (0,0)$ satisfies this requirement.

Suppose $A > 0$. If $B \geq 0$, then for $x \in \mathbb{Z}^+$, $C_{A,B}(x) \in \mathbb{Z}^+$, and so $0 \notin \langle x \rangle$. But if $B \leq 0$, then for $x \in \mathbb{Z}^-$, $C_{A,B}(x) \in \mathbb{Z}^-$, and so $0 \notin \langle x \rangle$. Thus, $A \leq 0$. Suppose $A = 0$ and $B \neq 0$. Then for odd x, $C_{0,B}(x) = B$. Thus (ignoring the trivial case), $A \neq 0$.

We can restrict A even further than $A < 0$. For $x \neq 0$, $C_{A,B}(x) = 0$ only if $x = -\frac{B}{A}$. Thus it must be the case that $A|B$. Now to contain 0, every trajectory must contain $-\frac{B}{A}$. Recall Theorem 3, which states that $1 \in \langle x \rangle$ only if $B = 1$. The proof showed that if $B \nmid 1$, then there exists an x such that $1 \notin \langle x \rangle$. Likewise, unless $B \,\Big|\, \left(-\frac{B}{A}\right)$, there exists an x such that $\left(-\frac{B}{A}\right) \notin \langle x \rangle$. Therefore, $A = -1$.

If $B < -1$, it is easily shown that that $B < x < 0$ implies $B < D_{-1,B}(x) < 0$. Similarly, if $B > 1$, it is easily shown that $0 < x < B$ implies $0 < D_{-1,B}(x) < B$. Thus, when $|B| > 1$, there are integers that never iterate to 0. Hence, $|B| = 1$. With that restriction on B, $\left|D_{-1,B}(x)\right| < |x|$ whenever $|x| > 1$, which makes it easy to prove that every trajectory does in fact converge to 0.

Therefore, when we consider the generalization for all integers, then the Collatz tree is connected if and only if $(A, B) \in \{(0,0), (-1,-1), (-1,1)\}$, which are all trivial cases. And so, as we suspected from the start, considering our generalization for all integers does not contribute to a better understanding of the 3x+1 problem, from either the trajectory or the tree perspective.

Summary of Findings

With our Ax+B generalization, we have proven several interesting facts for $A = 1$. First, every trajectory converges and x is in a loop if and only if $x \leq B$. Second, both the trajectory and the tree variations of the Collatz conjecture are satisfied if and only if $B = 1$. Yet the case of $(A, B) = (1,1)$ is sufficiently trivial that it does not contribute to our understanding of the 3x+1 Problem.

For $A > 3$, we proved that some trajectories do not contain 1 if $B \neq 1$, if $A \in \{5,181\}$, or if A is a Wieferich number. Yet we have nothing definitive to say regarding the tree variation of the conjecture when $A > 3$. Nevertheless, we showed that the expected value of the ratio of $T(x)$ to x is always greater than 1, which suggests that some divergent trajectories exist for all scenarios when $A > 3$.

When $A = 3$, we used stopping times and a proof by Terras [13,14] as evidence that every trajectory converges. Regarding the trajectory variation of the Collatz conjecture, we proved that it is possible for 1 to be in every trajectory only if $B = 1$. However, there are potentially other values of B that satisfy the tree variation of the conjecture, such as $B = 3$.

Conclusion

Since first proposed by Lother Collatz, the Collatz conjecture has been verified for increasingly many positive integers, yet still remains unproven. By looking at the 3x+1 problem under an Ax+B generalization, we understand a little better why every positive integer eventually iterates to 1. On the one hand, we demonstrate that $(A, B) = (3,1)$ appears to be the only non-trivial combination (since $A = 1$ was proven to be relatively trivia) such that every trajectory contains one. On the other hand, however, when we look at the problem from the Collatz tree perspective, there appear to be other non-trivial combinations of A and B such that the associated tree may be connected, such as $(A, B) = (3,3)$. In conclusion, we have shown why other cases of the generalization do not or likely do not share the conjectured properties of the 3x+1 function, but whether or not the 3x+1 function does in fact exhibit the conjectured properties is still a wide open question.

Appendix – Maple Code

Under $C_{3,3}(x)$, suppose $3 \in \langle x \rangle$ for all odd x less than some odd y. It is easy to see that if $x \in \langle y \rangle$ for any odd $x < y$, then $3 \in \langle y \rangle$. Thus, after verifying that $3 \in \langle x \rangle$ for $x \in \{1,3\}$, the following Maple 13 program was used to verify that $3 \in \langle x \rangle$ for all $x \leq N$. The variable 'divergeLimit' is some arbitrarily large number than prevents runtime errors if a trajectory does not appear to converge. The set 'trajectory' keeps track of the values in a trajectory in order to identify if a loop occurs that does not include 3. Note that the program will ignore values in a loop if it has already identified the loop's lowest value. On a 3.33 GHz processor with 4 MB of RAM, the runtime for $N = 10^8$ was about half an hour.

```
convergeA3B3:=proc(N)
  local diverge,divergeLimit,loop,i,x,trajectory;
  diverge:={};
  divergeLimit:=N^3;
  loop:={};
  for i from 5 to N by 2 do
    trajectory:={};
    x:=i;
    while x<divergeLimit and x>=i and evalb(x∉trajectory) do
      trajectory:=trajectory∪{x};
      x:=3*x+3;
      while mod(x,2)=0 do
        x:=x/2
      od;
    od;
    if x>=divergeLimit then
      diverge:=diverge∪{i}
    elif evalb(x ∈trajectory) then
      loop:=loop∪{x}
    fi;
  od;
  print("The following odd values less than N appear to diverge:",diverge);
  print("The following odd values less than N are in a loop:",loop)
end:
```

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