MTHE 227 Problem Set 11
Due Thursday December 01 2016 at the beginning of class

1. As a reminder, a torus with radii \( a \) and \( b \) is the surface of revolution of the circle \((x-b)^2 + z^2 = a^2\) in the \(xz\)-plane about the \(z\)-axis (\(a\) and \(b\) are positive real numbers, with \(b > a\)).

(For two pictures of a torus, see the last page of this problem set.)

(a) Find a function \( f(r, \theta, z) \) and a constant \( c \in \mathbb{R} \) so that the equation \( f(r, \theta, z) = c \) in cylindrical coordinates describes the torus with radii \( a \) and \( b \).

(b) Set up two triple integrals in cylindrical coordinates for the volume of the solid torus (the three-dimensional region bounded by a torus) with radii \( a \) and \( b \): one with order of integration \( dr \, dz \, d\theta \) and the other with order of integration \( dz \, dr \, d\theta \).

(c) Check that the volume of the solid torus is equal to \((\pi a^2)(2\pi b) = 2\pi^2 a^2 b\). (It is only necessary to integrate using one of the orders of part (b).)

(You may need to make a \( \sin / \cos \)-type trigonometric substitution.)

2. Find the volume of the region bounded by the surface \( z = \frac{1}{4}x^2 \), and the three planes \( y = 0, y = \ell \) and \( z = H \) in \( \mathbb{R}^3 \), as a function of \( \ell \) and \( H \).

3. Imagine a pool of still fluid (in other words, the fluid is static and in equilibrium). Let \( h \) denote the vertical coordinate, measured down from the surface of the fluid, and let \( x \) and \( y \) denote the usual Cartesian coordinates. As you likely know, if the fluid is incompressible (this is true of water, to a good approximation), the pressure of the fluid varies as\(^1\)

\[
p(h, y, z) = \delta gh,
\]

where \( \delta \) is the density of the fluid (assumed uniform), and \( g \) is the gravitational constant.

Because of the pressure difference at different heights, a region submerged in the fluid will have a net upward force on it, called the buoyant force, which may be computed as follows.

Let \( S \) be a closed (smooth, orientable) surface submerged in the fluid, bounding a region \( R \), and choose inward pointing normals. A small piece of \( S \) around the point \((x, y, h)\) with area \( \Delta A \) will have a force directed perpendicular to it and equal in magnitude (to a good approximation) to \( p(x, y, h) \Delta A \) (this is just the definition of pressure). To find its component directed up, we can compute the dot product

\[
-\mathbf{e}_h \cdot (p(x, y, h) \Delta A \mathbf{N}(x, y, h)) = (\delta gh \mathbf{e}_h) \cdot \mathbf{N}(x, y, h) \Delta A
\]

(the negative sign before \( \mathbf{e}_h \) is necessary because of the convention that \( h \) points down).

Defining the vector field

\[
\mathbf{B}(x, y, h) := (0, 0, -\delta gh) = -\delta gh \mathbf{e}_h,
\]

\(^1\)Instructor’s note: On the other hand, if you do not know why, and are curious why, ask me!
and taking $\Delta A \to 0$, the buoyant force on $S$ is therefore equal to the integral

$$\text{Buoyant Force} = \iint_S \mathbf{B} \cdot \hat{N} \, dS = \iint_S \mathbf{B} \cdot d\mathbf{S}.$$  

(a) Prove the following theorem, applying the divergence theorem:

**Theorem** (Archimedes). The buoyant force on $S$ is equal to the weight of the fluid displaced by $S$.

(Take care with the orientation of $\hat{N}$. In the statement, weight is the product of mass and the gravitational constant $g$.)

(b) Justify using (b): If $R$ is a region of uniform density $d$ placed in the pool, it will rise if $d < \delta$ and sink if $d > \delta$.

**Optional Problem.** Let $R$ be a ship modeled as a solid of the kind looked at in Problem 2, of mass $1,080,000$ kg, length $L = 30$ m and height $H = 10$ m. Take the fluid to be water (so, with density $\delta = 1,000$ kg/m$^3$). When the ship is floating at the surface of the water, where will the water level be (measured from the bottom of the ship)?

4. Let $R$ be the region $1 \leq x^2 + y^2 \leq 9$, $0 \leq z \leq 2$ in $\mathbb{R}^3$, and let $S$ be its boundary surface, oriented outward from $R$. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y, z) = (2x, xy^2, xyz).$$

(a) Sketch $R$. Notice that the boundary surface $S$ splits into four pieces.

(b) Compute the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ directly, by parametrizing each of the four pieces and computing the flux of $\mathbf{F}$ across each.

(c) Compute $\text{div} \mathbf{F}$, and compute the triple integral $\iiint_R \text{div} \mathbf{F} \, dV$ directly. The answer should be equal to that of part (b) by the divergence theorem.

5. As a reminder, spherical coordinates on $\mathbb{R}^3$ are given by the following map $D \to \mathbb{R}^3_{(x,y,z)}$:

$$x(\rho, \theta, \phi) = \rho \cos(\theta) \sin(\phi),$$

$$y(\rho, \theta, \phi) = \rho \sin(\theta) \sin(\phi),$$

$$z(\rho, \theta, \phi) = \rho \cos(\phi),$$

where

$$D = \{(\rho, \theta, \phi) \in \mathbb{R}^3 : \rho \geq 0, \ 0 \leq \theta \leq 2\pi, \ 0 \leq \phi \leq \pi\}.$$  

Check that

$$\det \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \det \begin{pmatrix} \partial x / \partial \rho & \partial x / \partial \theta & \partial x / \partial \phi \\ \partial y / \partial \rho & \partial y / \partial \theta & \partial y / \partial \phi \\ \partial z / \partial \rho & \partial z / \partial \theta & \partial z / \partial \phi \end{pmatrix} = \rho^2 \sin(\phi).$$
Just for fun (No need to hand-in). Back to the torus! We have seen that, parametrizing the generating circle of the torus with radii $a$ and $b$ by

$$t \mapsto (b + a \cos(t), a \sin(t)), \ t \in [0, 2\pi],$$

the torus may be parametrized by

$$\sigma: (\theta, t) \mapsto ((b + a \cos(t)) \cos(\theta), (b + a \cos(t)) \sin(\theta), a \sin(t)), \ \theta \in [0, 2\pi], \ t \in [0, 2\pi].$$

(This is likely a special case of the parametrization of the surface of revolution of a general parametrized curve that you found in Problem Set 9.)

Allow $\theta$ and $t$ in the parametrization of a torus to be arbitrary real numbers, disregarding the requirement that a parametrization of a surface be one-to-one in its interior.

Let $a = 1$ and $b = 2$. Write out the path $s \mapsto \sigma(2s, 3s), \ s \in [0, 2\pi]$ in Cartesian coordinates. How many times does this path wind around the $z$-axis as $s$ ranges from 0 to $2\pi$? How many times does it wind around the circle $x^2 + y^2 = 4, \ z = 0$?

This recovers the parametrization of the trefoil from the beginning of the term! Here are two views of this curve on the surface of a torus:

![Torus parametrization](image)

Taking other pairs of integers $(p, q)$ the paths $s \mapsto \sigma(ps, qs), \ s \in [0, 2\pi]$ define curves on the torus surface known as $(p,q)$-torus links (a link is a knot with possibly more than one connected piece).

Some things to ponder: What is the condition on the pair $(p, q)$ so that the $(p,q)$-torus link is a knot (in other words, has a single connected piece)? What will the path $s \mapsto \sigma(s, \sqrt{2}s), \ s \in \mathbb{R}$ look like?