1 (Great Circles). The intersection of a sphere with a plane passing through its center is called a *great circle*. Let \( \Gamma \) be the great circle that is the intersection of the plane \( x + y + z = 0 \) with the sphere \( x^2 + y^2 + z^2 = R^2 \) of radius \( R \) centered at the origin.

(a) Find a parametrization of \( \Gamma \). (*Suggestion:* Begin by finding two perpendicular vectors lying in the plane \( x + y + z = 0 \).)

(b) Check that the arclength of \( \Gamma \) is equal to \( 2\pi R \).

(c) If \( t \mapsto (x(t), y(t), z(t)) \) is a parametrization of \( \Gamma \), explain why \( t \mapsto (x(t), y(t), -z(t)) \) is a parametrization of the great circle \( \Gamma' \) cut out from the same sphere by \( x + y - z = 0 \).

*Optional Problem.* Find the points of intersection of \( \Gamma \) and \( \Gamma' \), and find the angle of intersection between the tangent lines to \( \Gamma \) and \( \Gamma' \) at these points.

2 (Velocity Perpendicular to Position). For a pair of parametrized paths \( q(t), r(t) \) in \( \mathbb{R}^3 \), show that

\[
\frac{d}{dt} (q(t) \cdot r(t)) = q'(t) \cdot r(t) + q(t) \cdot r'(t)
\]

(here \( \cdot \) denotes the dot product and \( ' \) the derivative with respect to \( t \)). Apply this identity to show the following: if the velocity of a parametrized path is always perpendicular to its position, then the curve traced out by the parametrization lies on the surface of a sphere.

3 (Flow Lines).  

(a) Check that the path \( t \mapsto (\cos(2t), \sin(2t)) \) is a flow line of the vector field \( \mathbf{F}(x, y) = (4y, -x) \) on \( \mathbb{R}^2 \). Sketch the vector field, the path, and check that the path is everywhere tangent to \( \mathbf{F} \).

(b) Find the flow lines of the vector field \( \mathbf{G}(x, y) = (1, -y^2) \), defined on the first quadrant \( \{(x, y) : x > 0, y > 0\} \) of \( \mathbb{R}^2 \). Which flow line passes through the point \((1, 1)\)?