Reminder. In lecture, we have defined the polar coordinate direction vector fields \( e_r \) and \( e_\theta \). These may be expressed in terms of the Cartesian direction vector fields (the latter also known as Cartesian direction vectors, the fields being constant) as

\[
\begin{align*}
e_r(x, y) &= \cos \theta e_x + \sin \theta e_y = \frac{x e_x + y e_y}{\sqrt{x^2 + y^2}}, \\
e_\theta(x, y) &= -\sin \theta e_x + \cos \theta e_y = \frac{-y e_x + x e_y}{\sqrt{x^2 + y^2}}.
\end{align*}
\]

Going the other way, we have

\[
\begin{align*}
e_x(r, \theta) &= \cos \theta e_r + \sin \theta e_\theta = \frac{r d\theta}{dt} e_r + \frac{d}{dt} e_\theta, \\
e_y(r, \theta) &= \sin \theta e_r + \cos \theta e_\theta = \frac{r}{\sqrt{r^2 + \theta^2}} e_r + \frac{d}{dt} e_\theta.
\end{align*}
\]

Intuitively, \( e_r \) and \( e_\theta \) are steps of unit length in the directions of increasing \( r \) and \( \theta \), respectively.

1 (Velocity and Acceleration in Polar Coordinates). We have seen that for a path parametrized in polar coordinates by \( t \mapsto (r(t), \theta(t)) \), \( t \in [a, b] \), the velocity and acceleration vectors are

\[
\begin{align*}
v(t) &= \frac{d}{dt} e_r(r(t), \theta(t)) + \frac{d}{dt} e_\theta(r(t), \theta(t)), \\
a(t) &= \left( \frac{d^2 r}{dt^2} - r \left( \frac{d \theta}{dt} \right)^2 \right) e_r(r(t), \theta(t)) + \left( r \frac{d^2 \theta}{dt^2} + 2 \frac{dr}{dt} \frac{d \theta}{dt} \right) e_\theta(r(t), \theta(t)).
\end{align*}
\]

To gain some understanding of the meaning of the various terms in the expression for the acceleration, for each of the following paths: sketch the path, compute the velocity and acceleration in polar coordinates, and sketch the velocity and acceleration vectors at a few points.

(a) (Accelerating Linear Motion) The path \( t \mapsto (t^2, \pi/4) \), \( t > 0 \).

(b) (Uniform Circular Motion) The path \( t \mapsto (R, 2016 t) \), \( t \in \mathbb{R} \). For this path, check that

\[
\|a(t)\| = \frac{\|v(t)\|^2}{R} \text{ for all } t.
\]

(This example is meant to shed some light on the \( -r(d\theta/dt)^2 \) term.)
(c) (Accelerating Circular Motion) The path \( t \mapsto (R, 1008t^2), \ t \in \mathbb{R}. \)

(d) (Archimedean Spiral) The path \( t \mapsto (t, t), \ t > 0. \)

(One can think of this example as the path followed by a ball rolling radially at unit speed on a platform rotating with unit angular speed, from the reference frame of someone not standing on the platform. It is one of the simplest examples in which the \( 2 \frac{dr}{dt} \frac{d\theta}{dt} \) term is nonzero.)

(e) (A Cardioid) The path \( t \mapsto (1 + \cos(t), t) = (r(t), \theta(t)), \ t \in [0, 2\pi]. \) This is one possible parametrization of the cardioid from Problem Set 5. Sketch the velocity and acceleration at \( t = 0, t = \pi/4, t = \pi/2 \) and \( t = \pi. \)