MTHE 227 Problem Set 9
Due Thursday November 17 2016 at the beginning of class

1 (Cross-Product in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \)). For this problem, to help distinguish between the cross-products in 2- and 3-space, for vectors

\[
\mathbf{v}_1 = (x_1, y_1), \quad \mathbf{v}_2 = (x_2, y_2) \text{ in } \mathbb{R}^2 \quad \text{and} \quad \mathbf{w}_1 = (x_1, y_1, z_1), \quad \mathbf{w}_2 = (x_2, y_2, z_2) \text{ in } \mathbb{R}^3,
\]
write

\[
cross_2(\mathbf{v}_1, \mathbf{v}_2) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \quad \text{and} \quad cross_3(\mathbf{w}_1, \mathbf{w}_2) = \begin{vmatrix} e_x & e_y & e_z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}.
\]

Embed \( \mathbb{R}^2_{(x,y)} \) into \( \mathbb{R}^3_{(x,y,z)} \) by the map \( (x, y) \mapsto (x, y, 0) \) (the image being the plane \( z = 0 \)).

(a) Let \( \mathbf{v}_1, \mathbf{v}_2 \) be vectors in \( \mathbb{R}^2_{(x,y)} \) and \( \mathbf{w}_1, \mathbf{w}_2 \) their images under the embedding. Check that

\[
cross_2(\mathbf{v}_1, \mathbf{v}_2) = cross_3(\mathbf{w}_1, \mathbf{w}_2) \cdot e_z.
\]

(b) Let \( \mathbf{r} : t \mapsto (x(t), y(t), 0), \ t \in [a,b] \) be a parametrized path in \( \mathbb{R}^3_{(x,y,z)} \) (thought of as the image of a parametrized path in \( \mathbb{R}^2_{(x,y)} \) under the above embedding). Denote the velocity vector at time \( t \) by \( \mathbf{r}'(t) = (x'(t), y'(t), 0) \). Check that

\[
\mathbf{n}_+(t) := (y'(t), -x'(t), 0) = cross_3(\mathbf{r}', e_z) \quad \text{and} \quad \mathbf{n}_-(t) := (-y'(t), x'(t), 0) = cross_3(e_z, \mathbf{r}').
\]

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Optional Problem (Harder). Embed \( \mathbb{R}^2_{(x,y)}, \mathbb{R}^2_{(y,z)} \) and \( \mathbb{R}^2_{(x,z)} \) into \( \mathbb{R}^3_{(x,y,z)} \) as the planes \( z = 0, \ x = 0 \) and \( y = 0 \), respectively. Let \( \pi_z : \mathbb{R}^3_{(x,y,z)} \rightarrow \mathbb{R}^2_{(x,y)} \) be the projection map \( (x, y, z) \mapsto (x, y) \), and similarly define \( \pi_x \), the projection onto \( \mathbb{R}^2_{(y,z)} \), and \( \pi_y \), the projection onto \( \mathbb{R}^2_{(x,z)} \).

Let \( P \) be a parallelogram in \( \mathbb{R}^3 \), and denote its images under the above projections by \( P_x = \pi_x(P), \ P_y = \pi_y(P) \) and \( P_z = \pi_z(P) \). Show that

\[
\text{area}(P) = \sqrt{\text{area}(P_x)^2 + \text{area}(P_y)^2 + \text{area}(P_z)^2}.
\]

Conclude, by applying the Cauchy-Schwarz inequality or otherwise, that

\[
\text{area}(P) \geq \frac{1}{\sqrt{3}} (\text{area}(P_x) + \text{area}(P_y) + \text{area}(P_z)) = \sqrt{3} \cdot \text{Arithmetic Mean}(\text{area}(P_x), \text{area}(P_y), \text{area}(P_z)).
\]

Can you find a \( P \) for which equality holds?
2 (Triple Cross Product). Find three vectors \( u, v, w \) in \( \mathbb{R}^3 \) such that
\[
(u \times v) \times w \neq u \times (v \times w).
\]
(If you are stuck, there is a suggestion at the end of the problem set. But try to find the vectors yourself — there are many possibilities.)

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Optional Problem (Messy). Show the identity
\[
(u \times v) \times w = (u \cdot w)v - (v \cdot w)u
\]
by expanding out in coordinates, and conclude that
\[
u \times (v \times w) = (u \cdot w)v - (u \cdot v)w.
\]
Conclude that \((u \times v) \times w = u \times (v \times w)\) if and only if either: \( u \) and \( w \) are both perpendicular to \( v \), or \( u = \lambda w \) for some \( \lambda \in \mathbb{R} \).
Also, conclude that
\[
u \times (v \times w) + v \times (w \times u) + w \times (u \times v) = 0 \quad \text{(the Jacobi identity)}.
\]

3 (Examples of Centroids of Curves). In lecture, we learned how to compute the coordinates of the center of mass of a curve \( C \) in \( \mathbb{R}^3 \). When \( C \) has uniform unit density (that is, \( \delta = 1 \)), the center of mass of \( C \) is also called the centroid. The coordinates of the centroid of \( C \) are then
\[
\frac{1}{\int_C ds} \left( \int_C x ds, \int_C y ds, \int_C z ds \right).
\]
A similar expression is true for a curve in \( \mathbb{R}^2 \), omitting the \( z \)-coordinate.

Find the centroids of the following curves in \( \mathbb{R}^2 \). You may use symmetry arguments to reduce the number of computations you need to do.

(a) The line segment parametrized by \( t \mapsto (t, mt) \), \( t \in [0, \frac{1}{m}] \), where \( m > 0 \) is the slope.

(b) The right semicircle \( t \mapsto (a \cos(t), a \sin(t)) \), \( t \in [-\frac{\pi}{2}, \frac{\pi}{2}] \) of radius \( a \) centered at the origin.

(c) The circle \( t \mapsto (b + a \cos(t), a \sin(t)) \), \( t \in [0, 2\pi] \) of radius \( a \) centered at \((b, 0)\), with \( b > a \) (feel free to write down the answer without computation if you see it).

(d) The piecewise curve \( C = C_1 + C_2 + C_3 \), where \( C_1 \) is the line segment from \((0, b)\) to \((a, b)\), \( C_2 \) the line segment from \((a, b)\) to \((a, -b)\), and \( C_3 \) the line segment from \((a, -b)\) to \((0, -b)\), where \( a > 0 \) and \( b > 0 \). The curve \( C \) is a \( a \times 2b \) rectangle, with the left side missing.

(e) Find the integral \( \int_C x ds \) for the parabola segment \( t \mapsto (t, t^2) \), \( t \in [0, 1] \).

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Optional Problem (Harder). Find the coordinates of the centroid of the parabola segment in part (e). The standard approach to the integrals involved uses sinh-substitution (!).
4 (Surfaces of Revolution). A surface of revolution is the surface obtained by rotating a plane curve \( C \) about a line \( \ell \) (called the axis of rotation) that is coplanar with \( C \).

To obtain a surface according to the definition in lecture, we require that \( \ell \) does not intersect \( C \), except possibly at the endpoints of \( C \). To obtain a smooth surface (except for at most finitely many nonsmooth curves, which do not affect surface area), we require that there exists a parametrization \( t \mapsto r(t) = (x(t), z(t)), \ t \in [a,b] \) of \( C \) with \( r'(t) \neq 0 \) for all \( t \) (with at most finitely many exceptions).

Suppose that \( C \) lies in the \( xz \)-plane with \( x > 0 \), \( \ell \) is the \( z \)-axis, and fix a parametrization of \( C \) as above.

(a) Find the unit vector that is obtained by rotating \( e_x \) counterclockwise by \( \theta \) radians about the \( z \)-axis.

(b) Using the parametrization \( t \mapsto r(t) = (x(t), z(t)), \ t \in [a,b] \) of \( C \), parametrize the curve obtained by rotating \( C \) counterclockwise by \( \theta \) radians about the \( z \)-axis (it will lie in the plane spanned by \( e_z \) and the vector from part (a)). Your parametrization will involve the functions \( x(t) \) and \( z(t) \).

(c) Parametrize the surface of revolution of \( C \), taking one of the parameters to be the parameter \( t \) of \( C \), and the other parameter to be the angle \( \theta \). What do the \( t \)- and \( \theta \)-coordinate curves look like?

(d) Find the tangent vectors \( T_t(t,\theta) \) and \( T_\theta(t,\theta) \) at all points.

(e) Find the normal \( N(t,\theta) = T_t(t,\theta) \times T_\theta(t,\theta) \) and its magnitude \( \|N(t,\theta)\| \) at all points.

(f) Show that the surface area of the surface of revolution of \( C \) is equal to

\[
2\pi \int_a^b x(t) \sqrt{x'(t)^2 + z'(t)^2} \, dt = 2\pi \int_C x \, ds.
\]

(g) Conclude that the following theorem holds:

**Theorem** (Pappus). The surface area of the surface of revolution of a curve \( C \) is equal to the product

\[
\text{arclength}(C) \cdot \text{distance travelled by the centroid of } C.
\]

(h) For each of the curves in Problem 3, sketch its surface of revolution about the \( z \)-axis and find the surface area using Pappus’s theorem.

Possibility for 2: \( u = (1,0,0), \ v = (1,0,0) \) and \( w = (0,1,0) \)