MATH 328
REAL ANALYSIS
WINTER 2010

Assignment 4
(due Tuesday, March 23, 2010)

(1) (a) Prove that the finite union and the finite intersection of compact sets (in a metric space) is compact.
   (b) Is the same true for infinite unions or intersections. Justify your answer by proofs or counter examples.
   (c) Prove Cantor's Intersection Theorem: A decreasing sequence of non-empty compacts subsets $A_1 \supset A_2 \supset \cdots$ of a metric space has non-empty and compact intersection.

(2) Let $F_1 = [0, 1/3] \cup [2/3, 1]$ be obtained from $[0, 1]$ by removing the middle third. Repeat, obtaining

$$F_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$

In general, F_n is a union of intervals and F_{n+1} is obtained by removing the middle third of these intervals. Let

$$C = \bigcap_{n=1}^{\infty} F_n, \qquad \text{the Cantor set.}$$

Prove:
(a) C is compact.
(b) C is uncountable.
(c) The interior of C is empty.
(d) C is totally disconnected; that is, if $x, y \in C$ and $x \neq y$, then $x \in U$ and $y \in V$ where U and V are open sets that disconnect C.

(3) (a) Prove that the Cantor set C has measure zero.
   (b) Prove that

$$\{x \in [0, 1] \mid x \text{ has no 5 as digit in its decimal expansion}\}$$

has measure zero.

(4) Prove that an open set $U \subset \mathbb{R}$ is the union of countably many open intervals.

(5) The goal of this problem is to prove the Dominated Convergence Theorem.
Suppose that f_n ($n \in \mathbb{N}$) and g are in $L^1[0,1]$ and that $|f_n| \leq g$ for all $n \in \mathbb{N}$. Also suppose that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e.

(a) Define, for $m < n$,

$$u_{mn} = \max(f_m, f_{m+1}, \dots, f_n)$$

and

$$l_{mn} = \min(f_m, f_{m+1}, \dots, f_n).$$

Show that $u_m = \lim_{n\to\infty} u_{mn}$ belongs to $L^1[0,1]$, as does $l_m = \lim_{n\to\infty} l_{mn}$.
Hint: Monotone Convergence Theorem

(b) Show that u_m is a decreasing sequence of functions converging to f almost everywhere, and l_m is an increasing sequence converging to f almost everywhere.

(c) Show that

$$\lim_{m\to\infty} \int u_m = \lim_{m\to\infty} \int l_m = \int f.$$

Hint: Monotone Convergence Theorem

(d) Prove the Dominated Convergence Theorem, i.e.,

$$\lim_{n\to\infty} \int f_n = \int f.$$

Hint: $l_n \leq f_n \leq u_n$.