MATH 328
REAL ANALYSIS
WINTER 2010

Assignment 3
(due Monday, March 1, 2010)

(1) Define a map $\mathcal{D}$ on $C[0,1]$ as follows:

$$\mathcal{D}f(x) = \begin{cases} \frac{2}{3} + \frac{1}{3}f(3x), & 0 \le x \le \frac{1}{3} \\ (2 + f(1))(\frac{2}{3} - x), & \frac{1}{3} \le x \le \frac{2}{3} \\ x - \frac{2}{3}, & \frac{2}{3} \le x \le 1. \end{cases}$$

(a) Sketch the graph of some function f and $\mathcal{D}f$.
(b) Show that $\mathcal{D}$ is a contraction.
(c) Describe the fixed point

(2) Show that the Hausdorff metric d_H for the space of compact subsets of $\mathbb{R}^n$, defined by

$$d_H(A, B) := \max\{\sup_{a \in A} \operatorname{dist}(a, B), \sup_{b \in B} \operatorname{dist}(b, A)\}$$

is actually a metric.

(3) (a) Show that $\rho(x, y) = |e^x - e^y|$ is a metric on $\mathbb{R}$.
(b) Let (X, ρ) be a metric space. Show that for a subset $A \subset X$ we have: A is closed if and only if $A^c := X \backslash A$ is open.

(4) (a) Show that the only connected subsets of $\mathbb{R}$ are intervals (which may be finite or infinite and may or may not include the endpoints).
(b) Show that a connected open set $U \subset \mathbb{R}^n$ is path connected. [Hint: Fix $u_0 \in U$. Consider the set V of points in U path connected to u_0, and the set W of points in U not path connected to u_0.]

(5) Let X be a sequentially compact metric space. Consider an open cover $\{U_i \mid i \in I\}$ of X. Show that there is an $r > 0$ such that for each $x \in X$, $B_r(x) \subset U_i$ for some $i \in I$.
(Note that this statement is needed to prove that sequential compactness implies compactness, thus you should not use the equivalent characterizations of sequential compactness for showing this!)