MATH 328
REAL ANALYSIS
WINTER 2010

Assignment 2
(due Monday, February 8, 2010)

(1) Let $(V, \|\cdot\|)$ be a normed vector space.

(a) Prove that

$$|\,\|x\| - \|y\|\,| \leq \|x - y\| \qquad \text{for all } x, y \in V$$

and use this to show that the mapping

$$\begin{aligned} \|\cdot\| : V &\to \mathbb{R} \\ x &\mapsto \|x\| \end{aligned}$$

is continuous.

(b) Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in V which converge to x and y, respectively. Show that the sequences $(x_n + y_n)_{n\in\mathbb{N}}$ and, for $\lambda \in \mathbb{R}$, $(\lambda x_n)_{n\in\mathbb{N}}$ converge to $x + y$ and λx, respectively.

(2) For $0 < p < \infty$, put $(\lambda_1, \ldots, \lambda_n \in \mathbb{R})$

$$\|(\lambda_1, \ldots, \lambda_n)\|_p := \sqrt[p]{|\lambda_1|^p + \cdots + |\lambda_n|^p}.$$

Show that $\|\cdot\|_p$ is, for $p \geq 1$, a norm on $\mathbb{R}^n$. What goes wrong in the case $0 < p < 1$?

(3) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $C[0,1]$ with $\|f_n\| \leq 3$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define a function F_n by

$$F_n(x) := \int_0^x f_n(t)dt \qquad (0 \leq x \leq 1).$$

Show that all F_n are elements in $C[0,1]$ and that the sequence $(F_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence.

(4) Let A be the 3×3-matrix

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 4 \end{pmatrix}.$$

Define a form $\langle \cdot, \cdot \rangle_A$ by

$$\langle x, y \rangle_A := xAy^T \qquad (x, y \in \mathbb{R}^3)$$

(where vectors x in $\mathbb{R}^3$ are written as rows, their transpose x^T is the corresponding column vector, and xAy^T means the usual matrix multiplication with rows and columns). Show that $\langle \cdot, \cdot \rangle_A$ is an inner product on $\mathbb{R}^3$.

(5) (a) Show that every inner product space V satisfies the parallelogram law:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in V$.

(b) Use this to show that the sup-norm on $C[0, 1]$ is not derived from an inner product $\langle \cdot, \cdot \rangle$ by $\|f\| = \sqrt{\langle f, f \rangle}$.