Introduction to Real Analysis

0. Language and Methods of Mathematics

(Quantifiers and Proofs)

0.1. Notations for numbers

$\mathbb{N}$ = natural numbers $\quad 1, 2, 3, 4, \ldots$

$\mathbb{Z}$ = integers $\quad -4, -3, -2, -1, 0, 1, 2, 3, \ldots$

$\mathbb{Q}$ = rational numbers $\quad \frac{p}{q} \ (p, q \in \mathbb{Z}, q \neq 0)$

$\mathbb{R}$ = real numbers

0.2. Quantifiers:

Mathematical statements are often of the form:

for all ... it is true that ...

there exists ... so that it is true that ..

Quantifiers: $\forall$ for all

$\exists$ there exists

Examples: $\forall x \in \mathbb{R}: (x+1)^2 = x^2 + 2x + 1$

$\exists x \in \mathbb{R}: x^2 + 2x + 1 = 0$

(indeed: for $x = -1$)

Note: 1) $\forall \neq \exists$, e.g.

$\forall x \in \mathbb{R}: x \geq 0$ false

$\exists x \in \mathbb{R}: x \geq 0$ true

2) Important to specify domain of quantifier

$\forall x \in \mathbb{R}: x^2 \geq x$ false

$\forall x \in \mathbb{N}: x^2 \geq x$ true

$\exists x \in \mathbb{Q}: x^2 = 2$ false

$\exists x \in \mathbb{R}: x^2 = 2$ true

3) $\forall \exists \neq \exists \forall$

$\forall$ person p $\exists$ person m : m is mother of p

"true"

("everybody has a mother")

$\exists$ person m $\forall$ person p : m is mother of p

"false"

("everybody has the same mother")

mathematical examples:

(i) $\forall y \in \mathbb{R} \ \exists x \in \mathbb{R} : x + y = 1$ true

(choose $x = 1 - y$),

(ii) $\exists x \in \mathbb{R} \ \forall y \in \mathbb{R} : x + y = 1$ false

note the difference:

in (i), x is allowed to depend on y

but in (ii), x must be choosen independently of y

(*) 4)

0.3. Negation of quantified sentences:

$S(x)$ statement depending on variable x

not $S(x)$ negation of that statement

e.g.: $S(x) : x^2 = x$

not $S(x) : x^2 \neq x$

$$\text{not}\,(\forall x \in \mathbb{R} : S(x)) \iff \exists x \in \mathbb{R} : \text{not}\ S(x)$$

It is not true that there is always snow on Christmas $\iff$ There is at least one Christmas without snow

$$\text{not}\,(\exists x \in \mathbb{R} : S(x)) \iff \forall x \in \mathbb{R} : \text{not}\ S(x)$$

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4) $\exists x_1 \exists x_2 \ \hat{=}\ \exists x_2 \exists x_1 \ \hat{=}\ \exists x_1, x_2$

$\forall x_1 \forall x_2 \ \hat{=}\ \forall x_2 \forall x_1 \ \hat{=}\ \forall x_1, x_2$

$$\forall x \in \mathbb{R}\ \forall y \in \mathbb{R} \quad x+y=y+x$$

$$\Updownarrow$$

$$\forall y \in \mathbb{R}\ \forall x \in \mathbb{R} \quad x+y=y+x$$

$$\Updownarrow$$

$$\forall x, y \in \mathbb{R} \quad x+y=y+x$$

$$\text{not } (\forall x \in \mathbb{R} : x^2 = x) \iff \exists x \in \mathbb{R} : x^2 \neq x$$

(indeed: $x = 2$)

"true"

$$\text{not } (\exists x \in \mathbb{Q} : x^2 = 2) \iff \forall x \in \mathbb{Q} ; x^2 \neq 2$$

there does not exist a rational number whose square is 2

"true"

$$\text{not } (\exists x \in \mathbb{R} \; \forall y \in \mathbb{R} : x + y = 1)$$

$$\Updownarrow$$

$$\forall x \in \mathbb{R} \; \exists y \in \mathbb{R} : x + y \neq 1$$

true

in general: $\text{not } (\exists x_1 \in \mathbb{R} \; \forall x_2 \in \mathbb{R} \; \exists x_3 \in \mathbb{R} : S(x_1, x_2, x_3))$

$$\Updownarrow$$

$$\forall x_1 \in \mathbb{R} \; \exists x_2 \in \mathbb{R} \; \forall x_3 \in \mathbb{R} : \text{not } S(x_1, x_2, x_3)$$

0.4. Proofs:

Proof = "sequence of arguments meant to convince us that some statement is true"

= "logically deduction of a theorem from the premises of the theorem, the axioms, or previously proved statements or theorems"

most theorems are of the form: $P \Rightarrow Q$

for statements P, Q

note: $P \Leftrightarrow Q$ means $P \Rightarrow Q$ and $Q \Rightarrow P$

0.5 Basic Techniques for Proofs:

1) Direct proof of $P \Rightarrow Q$

assume hypothesis P, use axioms, computations or other theorems and rules of logic to infer Q

"building a bridge of logical statements connecting P and Q

$$P \Rightarrow P_1 \Rightarrow P_2 \Rightarrow \dots \Rightarrow Q_2 \Rightarrow Q_1 \Rightarrow Q$$

2) Proof by contraposition of $P \Rightarrow Q$

note: $P \Rightarrow Q$ equivalent to $\underbrace{\text{not } Q \Rightarrow \text{not } P}_{\text{contraposition of } P \Rightarrow Q}$

[If it rains, the street becomes wet $\Leftrightarrow$ If the street is not wet, then it does not rain]

thus: prove $P \Rightarrow Q$ by actually proving

not $Q \Rightarrow$ not P

Example:

$x_1 + \dots + x_n \neq 0 \Rightarrow \exists i : x_i \neq 0$

is equivalent to

$$\underbrace{\text{not } (\exists i : x_i \neq 0)}_{\forall i : x_i = 0} \underset{\text{clear}}{\Rightarrow} \underbrace{\text{not } (x_1 + \dots + x_n \neq 0)}_{x_1 + \dots + x_n = 0}$$

3) Indirect proof = proof by contradiction

= reductio ad absurdum

we want to prove statement P

we assume not P and derive from this some contradiction (to hypothesis or known theorems)

thus: not P is false, i.e. P is true

(note: we use for this the principle of the excluded middle: always either P or not P, no third possibility)

Example: $\forall x > 0 : \frac{1}{x} > 0$

Proof: Assume $\underbrace{\text{not}\left(\forall x > 0 : \frac{1}{x} > 0\right)}_{\exists x > 0 :\ \underbrace{\text{not } \frac{1}{x} > 0}_{\frac{1}{x} \leq 0}}$

thus we assume: $x > 0$, $\frac{1}{x} \leq 0$

but then:

$\frac{1}{x} \leq 0 \Longrightarrow x \cdot \frac{1}{x} \leq 0 \cdot x$, i.e. $1 \leq 0$

multiplying with positive number x

contradiction

0.6. More specific techniques of proof

1) Proof by counterexamples

to conclude that statement of the form

$$\forall x \; S(x)$$

is false, it suffices to present one counterexample x s.th. $S(x)$ is false

2) Proof by cases

to prove $(P_1 \text{ or } P_2) \Rightarrow Q$

it suffices to show

$$P_1 \Rightarrow Q \quad \text{and} \quad P_2 \Rightarrow Q$$

3) Proof by induction for statement on natural numbers

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