Introduction to Real Analysis

0. Language and Methods of Mathematics
   (Quantifiers and Proofs)

0.1. Notations for numbers

\( \mathbb{N} = \text{natural numbers} \quad 1, 2, 3, 4, \ldots \)

\( \mathbb{Z} = \text{integers} \quad -4, -3, -2, -1, 0, 1, 2, 3, \ldots \)

\( \mathbb{Q} = \text{rational numbers} \quad \frac{p}{q} \quad (p, q \in \mathbb{Z}, \ q \neq 0) \)

\( \mathbb{R} = \text{real numbers} \)

0.2. Quantifiers: Mathematical statements are often of the form:

for all ... it is true that ... 

there exists... so that it is true that ...

Quantifiers: \( \forall \) for all \\
\( \exists \) there exists

Examples: \( \forall x \in \mathbb{R} \) : \((x+1)^2 = x^2 + 2x + 1\) \\
\( \exists x \in \mathbb{R} \) : \(x^2 + 2x + 1 = 0\) \\
   (Indeed: for \( x = -1 \))
Note: 1) $\forall \neq \exists$, e.g.

$\forall x \in \mathbb{R} : x \geq 0$  false 

$\exists x \in \mathbb{R} : x > 0$  true

2) Important to specify domain of quantifiers

$\forall x \in \mathbb{R} : x^2 \geq x$  false

$\forall x \in \mathbb{N} : x^2 \geq x$  true

$\exists x \in \mathbb{Q} : x^2 = 2$  false

$\exists x \in \mathbb{R} : x^2 = 2$  true

3) $\forall \neq \exists \forall$

$\forall$ person $p$  $\exists$ person $m$ : $m$ is mother of $p$  "true"  

("everybody has a mother")

$\exists$ person $m$  $\forall$ person $p$ : $m$ is mother of $p$  "false"

("everybody has the same mother")
mathematical examples:

(i) \( \forall y \in \mathbb{R} \ \exists x \in \mathbb{R} : x + y = 1 \) \quad true

\hspace{1cm} \text{(choose } x = 1-y)\),

(ii) \( \exists x \in \mathbb{R} \ \forall y \in \mathbb{R} : x + y = 1 \) \quad false

note the difference:

in (i), \( x \) is allowed to depend on \( y \)
but in (ii), \( x \) must be chosen independently of \( y \).

\( \star 4) \)

0.3. Negation of quantified sentences:

\( S(x) \) statement depending on variable \( x \)
not \( S(x) \) negation of that statement

e.g.: \( S(x) : x^2 = x \)
not \( S(x) : x^2 \neq x \)
not \( (\forall x \in \mathbb{R} : S(x)) \) \( \iff \) \( \exists x \in \mathbb{R} : \text{not } S(x) \)

It is not true that there is always snow on Christmas \((\iff)\)
There is at least one Christmas without snow

not \( (\exists x \in \mathbb{R} : S(x)) \) \( \iff \) \( \forall x \in \mathbb{R} : \text{not } S(x) \)
4) \( \exists x_1, \exists x_2 \Rightarrow \exists x_2 \exists x_1 \Rightarrow \exists x_1, x_2 \)

\( \forall x_1, \forall x_2 \Rightarrow \forall x_2 \forall x_1 \Rightarrow \forall x_1, x_2 \)

\( \forall x \in \mathbb{R} \forall y \in \mathbb{R} \quad x + y = y + x \)

\( \forall y \in \mathbb{R} \forall x \in \mathbb{R} \quad x + y = y + x \)

\( \forall x, y \in \mathbb{R} \quad x + y = y + x \)
\[ \neg (\forall x \in \mathbb{R} : x^2 = x) \iff \exists x \in \mathbb{R} : x^2 \neq x \quad (\text{indeed: } x = 2) \]

\[ \neg (\exists x \in \mathbb{R} : x^2 = 2) \iff \forall x \in \mathbb{R} : x^2 \neq 2 \]

There does not exist a rational number whose square is 2.

\[ \neg (\exists x \in \mathbb{R} \ \forall y \in \mathbb{R} : x + y = 1) \]

\[ \forall x \in \mathbb{R} \ \exists y \in \mathbb{R} : x + y \neq 1 \quad \text{true} \]

In general: \[ \neg (\exists x_1 \in \mathbb{R} \ \forall x_2 \in \mathbb{R} \ \exists x_3 \in \mathbb{R} : S(x_1, x_2, x_3)) \]

\[ \forall x_1 \in \mathbb{R} \ \exists x_2 \in \mathbb{R} \ \forall x_3 \in \mathbb{R} : \neg S(x_1, x_2, x_3) \]
0.4. Proofs:

Proof = "sequence of arguments meant to convince us that some statement is true"

= "logically deduction of a theorem from the premises of the theorem, the axioms, or previously proved statements on theorems"

Most theorems are of the form: $P \implies Q$

for statements $P, Q$

Note: $P \iff Q$ means $P \implies Q$ and $Q \implies P$

0.5. Basic Techniques for Proofs

1) Direct proof of $P \implies Q$

assume hypothesis $P$, use axioms, computations or other theorems and rules of logic to infer $Q$

"building a bridge of logical statements connecting $P$ and $Q$

$P \implies P_1 \implies P_2 \implies \ldots = \implies Q_2 \implies Q_1 \implies Q$
2) Proof by contraposition of \( P \implies Q \)

Note: \( P \implies Q \) equivalent to \( \neg Q \implies \neg P \) 

[If it rains, the street becomes wet]  
[If the street is not wet, then it does not rain]  

thus: prove \( P \implies Q \) by actually proving \( \neg Q \implies \neg P \) 

Example:

\[ x_1 + \ldots + x_n \neq 0 \implies \exists i : x_i \neq 0 \]

is equivalent to

\[ \forall i : x_i = 0 \implies x_1 + \ldots + x_n = 0 \]
3) Indirect proof = proof by contradiction = reductio ad absurdum

we want to prove statement P
we assume not P and derive from this some contradiction (to hypothesis or known theorem)
thus: not P is false, i.e. P is true

caveat: we use for this the principle of the excluded middle: always either P or not P (no third possibility)

Example: \( \forall x > 0 : \frac{1}{x} > 0 \)

Proof: Assume \( \neg(\forall x > 0 : \frac{1}{x} > 0) \)

\[ \exists x > 0 : \neg \left( \frac{1}{x} > 0 \right) \]

\[ \frac{1}{x} \leq 0 \]

thus we assume: \( x > 0 \), \( \frac{1}{x} \leq 0 \)

but then:

\[ \frac{1}{x} \leq 0 \rightarrow x \cdot \frac{1}{x} \leq 0 \cdot x \], i.e. \( 1 \leq 0 \)

contradiction
0.6. More specific techniques of proof

1) Proof by counterexamples
to conclude that statement of the form
   \( \forall x, S(x) \)
is false, it suffices to present one
counterexample \( x \) s.t. \( S(x) \) is false

2) Proof by cases
to prove \( (P_1 \lor P_2) \Rightarrow Q \)
it suffices to show
   \( P_1 \Rightarrow Q \) and \( P_2 \Rightarrow Q \)

3) Proof by induction for statement on
   natural numbers