MATH 891
GRADUATE CORE COURSE IN ANALYSIS I
FALL 2009

Assignment 1 (due September 28)

(1) Let $A_1, \dots, A_n$ be disjoint subsets of a set X. What is the smallest σ-algebra containing all sets $A_1, \dots, A_n$? Find all measures on this σ-algebra.
(2 points)

(2) Let $\mathfrak{A}$ be a σ-algebra and μ a measure on $\mathfrak{A}$. Prove the following statements:
(a) For all $A, B \in \mathfrak{A}$ we have
$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$$
(b) For all $A, B \in \mathfrak{A}$ we have
$$A \subset B \qquad \Longrightarrow \qquad \mu(A) \leq \mu(B)$$
(c) For all $A_n \in \mathfrak{A}$ ($n \in \mathbb{N}$) we have
$$\mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$$
(3 points)

(3) Let μ be a measure on a σ-algebra $\mathfrak{A}$ and consider $A_n \in \mathfrak{A}$ ($n \in \mathbb{N}$), $A \in \mathfrak{A}$.
(a) Show that $A_n \nearrow A$ implies
$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$
(b) Show that $A_n \searrow A$ and $\mu(A_1) < \infty$ implies
$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$
(c) Show that one cannot skip the assumption $\mu(A_1) < \infty$ in the previous part.
(3 points)

(4) Prove that if f is a real function on a measurable space X such that $\{x \mid f(x) \geq r\}$ is measurable for every rational r, then f is measurable.
(2 points)

(5) Consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1, & x \text{ rational} \end{cases}$$

(a) Recall how the Riemann integral is defined.
(b) Show that f is not Riemann integrable.
(c) Show that there exists an increasing sequence of functions $(f_n)_{n\in\mathbb{N}}$ such that f_n converges pointwise to f and such that, for all $n \in \mathbb{N}$, the Riemann integral of f_n is zero.

(3 points)