MATH 891
GRADUATE CORE COURSE IN ANALYSIS I
FALL 2009

Assignment 5 (due November 30)

(1) Let X and Y be Banach spaces. We define on the product space
$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$
a norm
$$\|(x, y)\| := \|x\| + \|y\|,$$
and denote by $X \oplus_1 Y$ the space $X \times Y$ equipped with this norm and componentwise addition and componentwise scalar multiplication. We will denote the elements (x, y) also by $x \oplus y$.
a) Show that $X \oplus_1 Y$ is a Banach space.
b) For a linear mapping $A : X \to Y$ we define its graph by
$$\text{graph}A := \{x \oplus Ax \mid x \in X\} \subset X \oplus_1 Y.$$
Show the closed graph theorem: If graph$A$ is closed in $X \oplus_1 Y$ then A is continuous.
(3 points)

(2) We use Baire's Theorem to show that there exists a continuous function on $[0, 1]$ which is not differentiable at any point in $[0, 1/2]$. We define, for each $n \in \mathbb{N}$, the set
$$F_n := \{f \in C[0, 1] \mid \text{there exists a point } x \in [0, 1/2] \text{ such that } \sup_{0<h<1/2} \frac{|f(x + h) - f(x)|}{h} \le n\ \}$$
(Note that the point x may depend on the function f.)
a) Show for each $n \in \mathbb{N}$: F_n is closed in $C[0, 1]$ and has no interior point.
b) Show with the help of (a) and Baire's Theorem that there exists a function $g \in C[0, 1]$ which is nowhere differentiable in $[0, 1/2]$.
(4 points)

(3) If μ is a measure, each $f \in L^\infty(\mu)$ defines a multiplication operator M_f on $L^2(\mu)$ into $L^2(\mu)$, such that

$$M_f(g) = fg.$$

Prove that $\|M_f\| \leq \|f\|_\infty$. For which measures μ is it true that $\|M_f\| = \|f\|_\infty$ for all $f \in L^\infty(\mu)$? For which $f \in L^\infty(\mu)$ does M_f map $L^2(\mu)$ onto $L^2(\mu)$?
(3 points)

(4) We denote, in the case $\nu \ll \mu$, the density h of the Radon-Nikodym theorem by $\frac{d\nu}{d\mu}$. In the following μ, ν, ω will be finite measures on a fixed σ-algebra. Show the following properties of the Radon-Nikodym derivatives:
  (a) If $\nu \ll \mu$ and f is a non-negative measurable function, then
  $$\int f d\nu = \int f \frac{d\nu}{d\mu} d\mu.$$
  (b) If $\nu \ll \mu$ and $\omega \ll \mu$, then also $(\nu + \omega) \ll \mu$ and we have
  $$\frac{d(\nu + \omega)}{d\mu} = \frac{d\nu}{d\mu} + \frac{d\omega}{d\mu}.$$
  (c) If $\nu \ll \mu \ll \omega$, then
  $$\frac{d\nu}{d\omega} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\omega}.$$
  (d) If $\nu \ll \mu$ and $\mu \ll \nu$, then
  $$\frac{d\nu}{d\mu} = \left(\frac{d\mu}{d\nu}\right)^{-1}.$$

(4 points)