Quantum groups and liberation of orthogonal matrix groups

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joint work with Teodor Banica
Liberation

group $\longrightarrow$ quantum group
Liberation

\[ \text{group} \rightarrow \text{quantum group} \]

\[ \text{orthogonal group} \rightarrow \text{quantum orthogonal group} \]

\[ \text{permutation group} \rightarrow \text{quantum permutation group} \]
Liberation

group $\rightarrow$ quantum group

orthogonal group $\rightarrow$ quantum orthogonal group

??? $\rightarrow$ quantum ???

permutation group $\rightarrow$ quantum permutation group
Orthogonal Hopf Algebra

is a \( C^* \)-algebra \( A \), given with a system of \( n^2 \) self-adjoint generators \( u_{ij} \in A \) \( (i, j = 1, \ldots, n) \), subject to the following conditions:

- The inverse of \( u = (u_{ij}) \) is the transpose matrix \( u^t = (u_{ji}) \).

- \( \Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \) defines a morphism \( \Delta : A \to A \otimes A \).

- \( \varepsilon(u_{ij}) = \delta_{ij} \) defines a morphism \( \varepsilon : A \to \mathbb{C} \).

- \( S(u_{ij}) = u_{ji} \) defines a morphism \( S : A \to A^{op} \).

These are compact quantum groups in the sense of Woronowicz.
Quantum Orthogonal Group (Wang 1995)

The quantum orthogonal group $A_o(n)$ is the universal unital $C^*$-algebra generated by $u_{ij}$ ($i, j = 1, \ldots, n$) subject to the relation

- $u = (u_{ij})_{i,j=1}^n$ is an orthogonal matrix

This means: for all $i, j$ we have

$$\sum_{k=1}^n u_{ik} u_{jk} = \delta_{ij} \quad \text{and} \quad \sum_{k=1}^n u_{ki} u_{kj} = \delta_{ij}$$
The quantum permutation group $A_s(n)$ is the universal unital $C^*$-algebra generated by $u_{ij} (i, j = 1, \ldots, n)$ subject to the relations

- $u_{ij}^2 = u_{ij} = u_{ij}^*$ for all $i, j = 1, \ldots, n$

- Each row and column of $u = (u_{ij})_{i,j=1}^n$ is a partition of unity:
  \[
  \sum_{j=1}^n u_{ij} = 1 \quad \sum_{i=1}^n u_{ij} = 1
  \]
  (this will feature prominently in the talk of Claus Koestler!)
How can we describe and understand intermediate quantum groups, sitting between these two cases:

\[ A_o(n) \rightarrow A \rightarrow A_s(n) \]
How can we describe and understand intermediate quantum groups, sitting between these two cases:

\[ A_0(n) \rightarrow A \rightarrow A_s(n) \]

Deal with quantum groups by looking on their representations!!!
Spaces of Intertwiners

Associated to an orthogonal Hopf algebra \((A, (u_{ij})_{i,j=1}^n)\) are the spaces of intertwiners:

\[
C_a(k, l) = \{ T : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l} \mid Tu^{\otimes k} = u^{\otimes l}T \}
\]

where \(u^{\otimes k}\) is the \(n^k \times n^k\) matrix \((u_{i_1j_1} \ldots u_{i_kj_k})_{i_1\ldots i_k ; j_1 \ldots j_k}\):

\[
u \in M_n(A) \quad \quad u : \mathbb{C}^n \to \mathbb{C}^n \otimes A
\]

\[
u^{\otimes k} : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes k} \otimes A^{\otimes k} \cong (\mathbb{C}^n)^{\otimes k} \otimes A
\]
Tensor Category with Duals

The collection of vector spaces $C_a(k, l)$ has the following properties:

- $T, T' \in C_a$ implies $T \otimes T' \in C_a$.

- If $T, T' \in C_a$ are composable, then $TT' \in C_a$.

- $T \in C_a$ implies $T^* \in C_a$.

- $id(x) = x$ is in $C_a(1, 1)$.

- $\xi = \sum e_i \otimes e_i$ is in $C_a(0, 2)$. 
Quantum Groups $\leftrightarrow$ Intertwiners

The compact quantum group $A$ can actually be rediscovered from its space of interwiners:

There is a one-to-one correspondence between:

- orthogonal Hopf algebras $A_o(n) \rightarrow A \rightarrow A_s(n)$

- tensor categories with duals $C_{ao} \subset C_a \subset C_{as}$. 
We denote by $P(k, l)$ the set of partitions of the set with repetitions $\{1, \ldots, k, 1, \ldots, l\}$. Such a partition will be pictured as

$$p = \{\begin{array}{c}
1 \ldots k \\
P \\
1 \ldots l
\end{array}\}$$

where $P$ is a diagram joining the elements in the same block of the partition.

Example in $P(5, 1)$:
Associated to any partition $p \in P(k, l)$ is the linear map

$$T_p : (\mathbb{C}^n)^\otimes k \to (\mathbb{C}^n)^\otimes l$$

given by

$$T_p(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_p(i, j) e_{j_1} \otimes \ldots \otimes e_{j_l}$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$.

$$T_-\{|\rangle\langle|\rangle\} (e_a \otimes e_b) = e_a \otimes e_b$$

$$T_-\{|\rangle\} (e_a \otimes e_b) = \delta_{ab} e_a \otimes e_a$$

$$T_-\{|\rangle\rangle\} (e_a \otimes e_b) = \delta_{ab} \sum_{cd} e_c \otimes e_d$$
Intertwiners of Quantum Permutation and of Quantum Orthogonal Group

Let \( NC(k, l) \subset P(k, l) \) be the subset of noncrossing partitions.

The tensor category of \( A_s(n) \) is given by:
\[
C_{as}(k, l) = \text{span}(T_p | p \in NC(k, l))
\]

The tensor category of \( A_o(n) \) is given by:
\[
C_{ao}(k, l) = \text{span}(T_p | p \in NC_2(k, l))
\]
Free Quantum Groups

A quantum group $A_o(n) \to \mathbf{A} \to A_s(n)$ is called free when its associated tensor category is of the form

$$C_{as} = \text{span}(T_p \mid p \in NC) \cup C_a \cup C_{ao} = \text{span}(T_p \mid p \in NC_2)$$
Free Quantum Groups

A quantum group $A_o(n) \to A \to A_s(n)$ is called free when its associated tensor category is of the form

$$C_{as} = \text{span}(T_p \mid p \in NC)$$

$$\cup$$

$$C_a = \text{span}(T_p \mid p \in NC_a),$$

$$\cup$$

$$C_{ao} = \text{span}(T_p \mid p \in NC_2)$$

for a certain collection of subsets $NC_a \subset NC$. 
Category of Noncrossing Partitions

A category of noncrossing partitions is a collection of subsets $NC_x(k, l) \subset NC(k, l)$, subject to the following conditions:

- $NC_x$ is stable by tensor product.
- $NC_x$ is stable by composition.
- $NC_x$ is stable by involution.
- $NC_x$ contains the “unit” partition $|\cdot|$.
- $NC_x$ contains the “duality” partition $\sqcap$.
Category of Noncrossing Partitions
↔ Free Quantum Groups

Let $NC_x$ be a category of noncrossing partitions, and $n \in \mathbb{N}$.

- $C_x = \text{span}(T_p | p \in NC_x)$ is a tensor category with duals.
- The associated quantum group $A_o(n) \to A \to A_s(n)$ is free.
- Any free quantum group appears in this way.
There are 6 Categories of Noncrossing Partitions:

\[
\begin{align*}
\{ \text{singletons and pairings} \} & \supset \{ \text{singletons and pairings (even part)} \} \\
\cap & \cap \\
\{ \text{all pairings} \} & \supset \{ \text{all partitions (even part)} \} \\
\cap & \cap \\
\{ \text{all partitions} \} & \supset \{ \text{with blocks of even size} \}
\end{align*}
\]
... and thus 6 free Quantum Groups:

\[
A_b(n) \leftarrow A_{b'}(n) \leftarrow A_o(n) \\
\downarrow \quad \downarrow \quad \downarrow \\
A_s(n) \leftarrow A_{s'}(n) \leftarrow A_h(n)
\]
• **Orthogonal**, if its entries are self-adjoint, and $uu^t = u^tu = 1$.

• **Magic**, if it is orthogonal, and its entries are projections.

• **Cubic**, if it is orthogonal, and $u_{ij}u_{ik} = u_{ji}u_{ki} = 0$, for $j \neq k$.

• **Bistochastic**, if it is orthogonal, and $\Sigma_j u_{ij} = \Sigma_j u_{ji} = 1$.

• **Magic’**, if it is cubic, with the same sum on rows and columns.

• **Bistochastic’**, if it is orthogonal, with the same sum on rows and columns.
More General Classification

\[ A_0(n) \rightarrow A_s(n) \]

\[ \downarrow \quad \downarrow \]

\[ C(O_n) \rightarrow C(S_n) \]
More General Classification

\[ A_o(n) \rightarrow A_{free} \rightarrow A_s(n) \]

\[ \downarrow \quad \downarrow \]

\[ C(O_n) \rightarrow C(S_n) \]

- There are exactly six free quantum groups \( A_{free} \)!
More General Classification

\[ A_o(n) \rightarrow A_{\text{free}} \rightarrow A_s(n) \]

\[ \downarrow \quad \downarrow \]

\[ C(O_n) \rightarrow C(G_{\text{easy}}) \rightarrow C(S_n) \]

- There are exactly six free quantum groups \( A_{\text{free}} \)!
- There are exactly six classical easy groups \( G_{\text{easy}} \)!
More General Classification

\[ A_0(n) \rightarrow A_{free} \rightarrow A_s(n) \]

\[ \downarrow \quad A_{easy} \quad \downarrow \]

\[ C(O_n) \rightarrow C(G_{easy}) \rightarrow C(S_n) \]

- There are exactly six free quantum groups \( A_{free} \)!
- There are exactly six classical easy groups \( G_{easy} \)!
- Can we have more easy quantum groups \( A_{easy} \)???
A quantum group satisfying \( A_o(n) \to A \to C(S_n) \) is called \textbf{easy} when its associated tensor category of intertwiners is spanned by partitions. The corresponding \textbf{full category of partitions} \( P_x \subset P \) must satisfy:

- \( P_x \) is stable by tensor product.
- \( P_x \) is stable by composition.
- \( P_x \) is stable by involution.
- \( P_x \) contains the “unit” partition \(|\).
- \( P_x \) contains the “duality” partition \(\sqcup\).
There are at least 3 more easy quantum groups

The following are full categories of partitions:

- \( P^*_o \): the set of pairings having the property that each string has an even number of crossings.

- \( P^*_b \): the set of singletons and pairings having the property that when removing the singletons, each string has an even number of crossings.

- \( P^*_{b'} \): the even part of \( P^*_b \), consisting of pairings having an even number of crossings, completed with an even number of singletons.
$P^*_o$ is generated by

The algebras $A^*_o(n), A^*_b(n), A^*_b'(n)$ are respectively the quotients of the algebras $A_o(n), A_b(n), A_b'(n)$ by the collection of relations

$$abc = cba$$

one for each choice of $a, b, c$ in the set $\{u_{ij}|i, j = 1, \ldots, n\}$. 
Literature

