MATH 891
GRADUATE CORE COURSE IN ANALYSIS I
FALL 2009

Assignment 2 (due October 14)

(1) Show by an example that we can have strict inequality in Fatou's Lemma.
(2 points)

(2) Suppose μ is a measure on X, $f : X \to [0, \infty]$ is measurable, $\int_X f d\mu = c$, where $0 < c < \infty$, and α is a constant. Prove that

$$\lim_{n\to\infty} \int_X n \log[1 + (f/n)^\alpha] d\mu = \begin{cases} \infty, & \text{if } 0 < \alpha < 1 \\ c, & \text{if } \alpha = 1 \\ 0, & \text{if } 1 < \alpha < \infty \end{cases}$$

(3 points)

(3) Let

$\mathfrak{A} := \{A \subset \mathbb{R} \mid A \text{ is a union of finitely many disjoint intervals}\}$

and

$$\lambda : \mathfrak{A} \to [0, \infty]$$

defined as follows: If $A = \bigcup_{k=1}^n I_k$ is a decomposition of A into finitely many disjoint intervals, then

$$\lambda(A) := \sum_{k=1}^n \text{length}(I_k).$$

Prove that $\mathfrak{A}$ is an algebra and that λ is a measure on the algebra $\mathfrak{A}$.
(3 points)

(4) (a) Show that the outer measure λ^* (and thus also the Lebesgue measure λ) is invariant under translations: for a set $A \subset \mathbb{R}$ and any $x \in \mathbb{R}$ we have that

$$\lambda^*(A + x) = \lambda^*(A) \qquad \text{where} \qquad A + x := \{a + x \mid a \in A\}.$$

(b) Let $X = (0,1] \cap \mathbb{Q}$ and $\mathfrak{A}$ the algebra of finite unions of intervals of the form $(a,b] \cap \mathbb{Q}$ (with $0 < a < b \leq 1$, $a, b \in \mathbb{Q}$), and μ a measure on $\mathfrak{A}$ defined by $\mu(a,b] = \infty$ for all intervals as above and $\mu(\emptyset) = 0$. Show that the extension of μ to the smallest σ-algebra containing $\mathfrak{A}$ is not unique.

(4 points)

(5) Let c_0 be the vector space of sequences of complex numbers which converge to zero. Determine directly (i.e., without using the Riesz representation theorem) all positive linear functionals $L : c_0 \to \mathbb{C}$. Prove your claims.

(3 points)