1. [Let \( X \) and \( Y \) be two independent random variables . . . ]

Since \( X \) and \( Y \) are independent and exponentially distributed with parameter \( \lambda \), their joint pdf is given by

\[
f(x, y) = f_X(x)f_Y(y) = \begin{cases} 
\lambda e^{-\lambda x} \lambda e^{-\lambda y} & \text{if } x, y \geq 0 \\
0 & \text{otherwise.}
\end{cases}
\]

(a) 

\[
E[U] = E[\min(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \min(x, y) f(x, y) \, dx \, dy \\
= \int_{0}^{\infty} \int_{0}^{\infty} \min(x, y) (\lambda^2 e^{-\lambda(x+y)}) \, dx \, dy \\
= \int_{0}^{\infty} \int_{0}^{y} x (\lambda^2 e^{-\lambda(x+y)}) \, dx \, dy + \int_{0}^{\infty} \int_{y}^{\infty} y (\lambda^2 e^{-\lambda(x+y)}) \, dx \, dy \\
= \int_{0}^{\infty} \int_{0}^{y} x (\lambda^2 e^{-\lambda(x+y)}) \, dx \, dy + \int_{0}^{\infty} \int_{0}^{x} y (\lambda^2 e^{-\lambda(x+y)}) \, dy \, dx \\
= \frac{1}{4\lambda} + \frac{1}{4\lambda} = \frac{1}{2\lambda}.
\]

\[
E[V] = E[\max(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \max(x, y) f(x, y) \, dx \, dy \\
= \int_{0}^{\infty} \int_{0}^{\infty} \max(x, y) (\lambda^2 e^{-\lambda(x+y)}) \, dx \, dy \\
= \int_{0}^{\infty} \int_{0}^{y} y (\lambda^2 e^{-\lambda(x+y)}) \, dx \, dy + \int_{0}^{\infty} \int_{y}^{\infty} x (\lambda^2 e^{-\lambda(x+y)}) \, dx \, dy \\
= \int_{0}^{\infty} \int_{0}^{y} y (\lambda^2 e^{-\lambda(x+y)}) \, dx \, dy + \int_{0}^{\infty} \int_{0}^{x} x (\lambda^2 e^{-\lambda(x+y)}) \, dy \, dx \\
= \frac{3}{4\lambda} + \frac{3}{4\lambda} = \frac{3}{2\lambda}.
\]

(b) We will do this part independently of the calculations in part (a). Note that since \( X \) and \( Y \) are both exponentially distributed with parameter \( \lambda \), we have \( E[X] = E[Y] = 1/\lambda \).

Observe that \( U + V = X + Y \) and \( UV = XY \), since for any real numbers \( x, y \), we have

\[
\min(x, y) + \max(x, y) = x + y \quad \text{and} \quad \min(x, y) \cdot \max(x, y) = xy.
\]
Therefore, $E[U + V] = E[X + Y] \overset{(1)}{=} E[X] + E[Y] = 2/\lambda$, the equality labeled (1) being a consequence of the linearity of expectation.

$E[UV] = E[XY] \overset{(2)}{=} E[X]E[Y] = 1/\lambda^2$, the equality labeled (2) being a consequence of the independence of $X$ and $Y$.

2. [Ghahramani, 8.3, # 1]

We are given

$$p(x, y) = \begin{cases} \frac{1}{25}(x^2 + y^2) & \text{if } x = 1, 2, \quad y = 0, 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

To find $p_{X|Y}(x|y)$, we first need to determine $p_Y(y)$:

$$p_Y(y) = \sum_x p(x, y) = \begin{cases} \frac{1}{25} \sum_{x=1}^2 (x^2 + y^2) = \frac{1}{25}(5 + 2y^2) & \text{if } y = 0, 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for $y = 0, 1, 2$, we have

$$p_{X|Y}(x|y) = \frac{p(x, y)}{p_Y(y)} = \begin{cases} \frac{x^2 + y^2}{5 + 2y^2} & \text{if } x = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$P(X = 2 \mid Y = 1) = p_{X|Y}(2|1) = \frac{2^2 + 1^2}{5 + 2(1)^2} = 5/7.$$\[17/86\]

Finally,

$$E[X \mid Y = 1] = \sum_x x p_{X|Y}(x|1) = \sum_{x=1}^2 x \left(\frac{x^2 + 1}{7}\right) = 1\left(\frac{2}{7}\right) + 2\left(\frac{5}{7}\right) = \frac{12}{7}.$$\[17/86\]

3. [Ghahramani, 8.3, # 4]

We have

$$P(1/4 < X < 1/2 \mid Y = 3/4) = \int_{1/4}^{1/2} f_{X|Y}(x \mid 3/4) \, dx$$

$$= \int_{1/4}^{1/2} \frac{3(x^2 + (3/4)^2)}{3(3/4)^2 + 1} \, dx$$

$$= \int_{1/4}^{1/2} \frac{1}{43}(48x^2 + 27) \, dx$$

$$= \frac{17}{86}.$$
4. [Ghahramani, 8.3, # 19]

(a) The triangle is bounded by the lines $x = 0$, $y = 0$, and $x + y = 1$. Since the area of the triangle is $1/2$ the joint pdf is given by
\[
f(x, y) = \begin{cases} 
2 & \text{if } x \geq 0, y \geq 0, x + y \leq 1, \\
0 & \text{otherwise}.
\end{cases}
\]

(b) For all $y \in [0, 1]$,
\[
f_Y(y) = \int_{-\infty}^{\infty} f(x, y) \, dx = \int_{0}^{1-y} 2 \, dx = 2(1 - y)
\]
and for all $y \in [0, 1)$,
\[
f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \begin{cases} 
\frac{1}{1-y} & \text{if } 0 \leq x \leq 1 - y, \\
0 & \text{otherwise}.
\end{cases}
\]
This means that conditional distribution of $X$ given $Y = y$ is uniform on the interval $[0, 1 - y]$, which in turn implies
\[
E(Y|Y = y) = \frac{1 - y}{2} \quad \text{if } 0 \leq y < 1.
\]

5. [Ghahramani, 8.4, # 6]

The joint pdf of $X$ and $Y$ is
\[
f_{X,Y}(x, y) = \begin{cases} 
\frac{1}{x^2} & \text{if } x \geq 1, y \geq 1, \\
0 & \text{otherwise}.
\end{cases}
\]

We are looking for the joint pdf of
\[
U = \frac{X}{Y} \quad \text{and} \quad V = XY.
\]

Let $R = \{(x, y) : x \geq 1, y \geq 1\}$ be the region where $f_{X,Y}(x, y)$ is positive. For $(x, y) \in R$ the pair of equations
\[
u = \frac{x}{y}, \quad v = xy
\]
has the unique solution
\[
x = \sqrt{uv}, \quad v = \sqrt{\frac{v}{u}}.
\]
Note that since $x, y > 0$, we have $u, v > 0$. Also, $x \geq 1$ is equivalent to $\sqrt{uv} \geq 1$, which holds if and only if $v \geq \frac{1}{u}$. Similarly, $y \geq 1$ is equivalent to $\sqrt{\frac{v}{u}} \geq 1$, which in turn holds if and only if $v \geq u$. Thus $h_1(x, y) = \frac{x}{y}$ and $h_2(x, y) = xy$ transform the region $R$ onto the region
\[
G = \{(u, v) : u > 0, v > 0, v \geq \frac{1}{u}, v \geq u\}.
\]
(sketch this region). On $G$ the mapping $(h_1, h_2)$ has inverse
\[ g_1(u,v) = \sqrt{uv}, \quad g_2(u,v) = \sqrt{\frac{v}{u}}. \]
All of these functions have continuous first-order partial derivatives on their domain, so we have
\[ f_{U,V}(u,v) = f_{X,Y}(g_1(u,v), g_2(u,v))|J(u,v)|, \quad \text{for } (u,v) \in G \]
where $J(u,v)$ is the Jacobian of $(g_1, g_2)$, i.e., the determinant of the matrix
\[ \begin{bmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \end{bmatrix}. \]
We have
\[ \frac{\partial g_1}{\partial u} = \frac{\partial \sqrt{uv}}{\partial u} = \frac{1}{2} \sqrt{\frac{v}{u}}, \quad \frac{\partial g_1}{\partial v} = \frac{\partial \sqrt{uv}}{\partial v} = \frac{\sqrt{u}}{v} \]
and
\[ \frac{\partial g_2}{\partial u} = \frac{\partial \sqrt{\frac{v}{u}}}{\partial u} = -\frac{1}{2} \frac{\sqrt{v}}{u^{3/2}}, \quad \frac{\partial g_1}{\partial v} = \frac{\partial \sqrt{\frac{v}{u}}}{\partial v} = \frac{1}{2} \frac{1}{\sqrt{uv}}. \]
Therefore
\[ |J(u,v)| = \frac{1}{2u} \neq 0 \]
and we conclude
\[ f_{U,V}(u,v) = \frac{1}{(\sqrt{uv})^2 \left(\frac{2}{\sqrt{2}}\right)^2} \frac{1}{2u} = \frac{1}{2uv^2}, \quad (u,v) \in G. \]

6. [Let $X$ and $Y$ be independent normal variables . . .]

Since $X$ and $Y$ are independent, the pdf of $Z$ is the convolution of $f_X$ and $f_Y$:
\[ f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y) \, dy \]
\[ = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(z-y)^2/2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2/2\sigma^2} \, dy \]
\[ = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^2 \int_{-\infty}^{\infty} e^{-\frac{(z-y)^2}{2\sigma^2}} - \frac{y^2}{2\sigma^2} \, dy. \]
We have
\[ \frac{(z-y)^2}{2\sigma^2} - \frac{y^2}{2\sigma^2} = -\frac{z^2 - 2zy + 2y^2}{2\sigma^2} = -\frac{z^2/2 + z^2/2 - 2zy + 2y^2}{2\sigma^2} \]
\[ = -\frac{z^2}{4\sigma^2} - \frac{(\sqrt{2}y - z/\sqrt{2})^2}{2\sigma^2} \]
\[ = -\frac{z^2}{4\sigma^2} - \frac{2(y - z/2)^2}{2\sigma^2} \]
\[ = -\frac{z^2}{4\sigma^2} - \frac{(y - z/2)^2}{\sigma^2}. \]
Thus
\[
f_Z(z) = \frac{1}{\sqrt{2\pi(2\sigma^2)}} e^{-\frac{z^2}{4\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(\sigma^2/2)}} e^{-\frac{(y-z/2)^2}{2(\sigma^2/2)}} dy.
\]

Now note that the integrand is the pdf of a normal random variable with mean \( \mu = z/2 \) and variance \( \sigma^2/2 \) and therefore its integral is
\[
\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(\sigma^2/2)}} e^{-\frac{(y-z/2)^2}{2(\sigma^2/2)}} dy = 1
\]
proving that
\[
f_Z(z) = \frac{1}{\sqrt{2\pi(2\sigma^2)}} e^{-z^2/4\sigma^2},
\]
so that \( Z \sim N(0, 2\sigma^2) \) as claimed.