1. [Two dice are thrown...]

The sample space is all pairs of integers between 1 and 6: \( S = \{(i, j) : i, j \in \{1, \ldots, 6\}\} \).

(a) \( EF \) is the event

\[
EF = \{(i, j) \in S : i + j \text{ is odd, } i = 1 \text{ or } j = 1\} = \{(1, 2), (1, 4), (1, 6), (2, 1), (4, 1), (4, 6)\}.
\]

(b) Since \( G \subset E \), we have \( E \cup G = E \).

(c) \( FG \) is the event that the sum is 5 and at least one die shows 1. Thus

\[
FG = \{(1, 4), (4, 1)\}.
\]

(d) \( EF^c \) is the event that the sum is odd and neither die shows 1, so

\[
EF = \{(2, 3), (2, 5), (3, 2), (3, 4), (3, 6), (4, 3), (4, 5), (5, 2), (5, 4), (5, 6)\}.
\]

(e) Since \( G \subset E \), we have \( EFG = FG = \{(1, 4), (4, 1)\} \) (by part (c)).

2. [Ghahramani, 1.2, # 6]

We will represent a quarter as “q”, a nickel as “n”, a dime as “d”, and a penny as “p”. Since it is not specified that the order in which the coins are drawn matters, we will assume that the order does not matter.

The sample space is thus

\[
S = \{\{q, q\}, \{q, n\}, \{q, d\}, \{q, p\}, \{n, n\}, \{n, d\}, \{n, p\}, \{d, p\}, \{p, p\}\}
\]

(a) The only way of obtaining 26 cents by drawing two of the given coins is if the two coins drawn are a quarter (25 cents) and a penny (1 cent). So, the event of drawing exactly 26 cents is simply \( \{\{q, p\}\} \).

(b) The event that more than 9 cents, but less than 25 cents, are drawn is \( \{\{n, n\}, \{n, d\}, \{d, p\}\} \).

(c) There is no way of getting exactly 29 cents using just two of the coins given. Hence, this event is the empty set, \( \emptyset \).
Remark:  This problem can also be attempted under the assumption that the order in which the
coins are drawn is noted. In such a case, the sample space becomes
\[ S = \{qq, qn, nq, dq, qp, pq, nn, dn, np, pn, dp, pd, pp\} \]
and the representation of events changes correspondingly.

3. [Let \( E, F, \) and \( G \) be...]

(a) The event that exactly \( E \) and \( F \) occur (but not \( G \)) is \( EFG^c \). It is easy to see that the event
that exactly two of \( E, F, \) and \( G \) occur is
\[
EFG^c \cup EGF^c \cup FGE^c = (EF - EFG) \cup (EG - EFG) \cup (FG - EFG)
\]
\[
= EF(EFG)^c \cup EG(EFG)^c \cup EG(EFG)^c
\]
\[
= (EF \cup EG \cup FG)(EFG)^c \quad \text{by the distributive property}
\]
\[
= (EF \cup EG \cup FG) - EFG.
\]

(b) Since \( EFG \subset EF \cup EG \cup FG \), we have
\[
P(A) = P((EF \cup EG \cup FG) - EFG) = P(EF \cup EG \cup FG) - P(EFG).
\]
Using the inclusion-exclusions formula for three events
\[
P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)
\]
we obtain
\[
P(EF \cup EG \cup FG)
\]
\[
= P(EF) + P(EG) + P(FG) - P(EFG) - P(EFG) - P(EGFG) + P(EFEGFG)
\]
\[
= P(EF) + P(EG) + P(FG) - 2P(EFG)
\]
so that
\[
P(A) = P(EF) + P(EG) + P(FG) - 3P(EFG).
\]

4. [Show the following generalization of the union bound...]
We know that for any two events \( A_1 \) and \( A_2 \),
\[
P(A_1 \cup A_2) \leq P(A_1) + P(A_2).
\]
The general statement can be shown using an induction argument:

\[
P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1 \cup (A_2 \cup \cdots \cup A_n)) \\
\leq P(A_1) + P(A_2 \cup \cdots \cup A_n) \\
= P(A_1) + P(A_2 \cup (A_3 \cup \cdots \cup A_n)) \\
\leq P(A_1) + P(A_2) + P(A_3 \cup \cdots \cup A_n) \\
\vdots \\
\leq P(A_1) + P(A_2) + \cdots + P(A_n).
\]

5. [A fair die is thrown twice...]

The sample space is \( S = \{(i, j) : i, j \in \{1, \ldots, 6\}\} \). This is a sample space with equally likely outcomes, so that each outcome has probability \( \frac{1}{|S|} = \frac{1}{36} \).

(a) Let \( A = \) "six turns up exactly once." Then

\[
A = \{(6, 1), \ldots, (6, 5), (1, 6), \ldots, (5, 6)\}
\]

and

\[
P(A) = \frac{|A|}{36} = \frac{10}{36} = \frac{5}{18}.
\]

(b) If \( B = \) "both numbers are odd," then \( B \) consists of all pairs of odd integers between 1 and 6, so that

\[
P(B) = \frac{|B|}{36} = \frac{3 \times 3}{36} = \frac{9}{36} = \frac{1}{4}.
\]

(c) If \( C = \) "sum is 4," then \( C = \{(1, 3), (2, 2), (3, 1)\} \) and

\[
P(C) = \frac{|C|}{36} = \frac{3}{36} = \frac{1}{12}.
\]

(c) Let \( D = \) "sum is divisible by 3." The possible outcomes for the sum are the integers 2, 3, \ldots, 12. Among these 3, 6, 9, and 12 are divisible by 3. Thus \( D \) is the union of the disjoint events: \( D = A_3 \cup A_6 \cup A_9 \cup A_{12} \), where \( A_i \) is the event that the sum is \( i \). Therefore

\[
P(A) = P(A_3 \cup A_6 \cup A_9 \cup A_{12}) = P(A_3) + P(A_6) + P(A_9) + P(A_{12})
\]

where

\[
P(A_3) = \frac{|A_3|}{36} = \frac{2}{36}, \quad P(A_6) = \frac{|A_6|}{36} = \frac{5}{36}, \quad P(A_9) = \frac{|A_9|}{36} = \frac{4}{36}, \quad P(A_{12}) = \frac{|A_{12}|}{36} = \frac{1}{36}.
\]

We obtain

\[
P(D) = \frac{2}{36} + \frac{5}{36} + \frac{4}{36} + \frac{1}{36} = \frac{3}{36} = \frac{1}{3}.
\]
(a) Let \( A \) denote the set of students taking physics and \( B \) the set of students taking chemistry. We want to know \( |A \cup B| \), the number of students in \( A \cup B \). If \( N \) denotes the number of grade 8 students and a student is randomly chosen from among these \( N \) students, then the probability that this student is in \( A \cup B \) is

\[
P(A \cup B) = \frac{|A \cup B|}{N}.
\]

From the identity

\[
P(A \cup B) = P(A) + P(B) - P(AB)
\]

we obtain

\[
P(A \cup B) = \frac{|A \cup B|}{N} = \frac{|A|}{N} + \frac{|B|}{N} - \frac{|AB|}{N}
\]

so that

\[
|A \cup B| = |A| + |B| - |AB| = 20 + 15 - 4 = 31.
\]

(b) Let \( C \) denote the set of students who take exactly one of physics and chemistry. Then \( C = (A \cup B) - AB \). Since \( AB \subset A \cup B \),

\[
P(C) = P((A \cup B) - AB) = P(A \cup B) - P(AB)
\]

so that

\[
|C| = |(A \cup B) - AB| = |A \cup B| - |AB|
\]

Thus from part (a) we obtain

\[
|C| = |A| + |B| - |AB| - |AB| = |A| + |B| - 2|AB| = 20 + 15 - 8 = 27.
\]

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**Bonus question**

We use induction on \( n \). Let \( n = 2 \). From class we know that for any two events \( A \) and \( B \),

\[
P(A \cup B) = P(A) + P(B) - P(AB).
\]

(*)

Therefore, if \( P(AB) = 0 \), we obtain \( P(A \cup B) = P(A) + P(B) \) as claimed.

Now let \( n > 2 \) and assume that for any collection \( B_1, \ldots, B_{n-1} \) of \( n - 1 \) events with \( P(B_iB_j) = 0 \) for \( i \neq j \), we have

\[
P\left( \bigcup_{i=1}^{n-1} B_i \right) = \sum_{i=1}^{n-1} P(B_i).
\]

(**)
Let $A_1, \ldots, A_n$ be events such that $P(A_i A_j) = 0$ for $i \neq j$. Then, letting $A = A_1$ and $B = A_2 \cup \cdots \cup A_n$, we have
\[ \bigcup_{i=1}^{n} A_i = A \cup B \]
and
\[ AB = A_1(A_2 \cup \cdots \cup A_n) = A_1 A_2 \cup A_1 A_3 \cup \cdots \cup A_1 A_n \]
so that by the union bound (Problem 4)
\[ P(AB) \leq \sum_{i=2}^{n} P(A_1 A_i) = 0. \]
Thus from (*) we obtain
\[ P\left( \bigcup_{i=1}^{n} A_i \right) = P(A \cup B) = P(A) + P(B) = P(A_1) + P(A_2 \cup \cdots \cup A_n). \]
Letting $B_i = A_{i+1}$ for $i = 1, \ldots, n-1$, the induction hypothesis (**) gives
\[ P(A_2 \cup \cdots \cup A_n) = P(A_2) + \cdots + P(A_n). \]
Combining the last two equations, we get
\[ P\left( \bigcup_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} P(A_i) \]
as claimed.