1. (a) An event $A$ can be independent of itself if and only if we have $P(A \cap A) = P(A)P(A)$, or equivalently,

$$P(A) = [P(A)]^2.$$ 

This is possible if and only if $P(A) = 0$ or $P(A) = 1$. Thus, an event $A$ can be independent of itself, and this can happen precisely when $P(A) = 0$ or $P(A) = 1$.

(b) No, this is not true. For example, in the experiment of throwing two fair dice, let $A = \text{"sum is 7,"}$ $B = \text{"first die shows 4,"}$ and $C = \text{"second die shows 3."}$ We have seen in class that $P(A) = P(B) = P(B) = 1/6$ and

$$P(AB) = P(AC) = \frac{1}{36}$$

so $A$ and $B$ are independent, and $A$ and $C$ are independent. However, $B - C = BC^c$ is the event that the first die shows 4 and the second shows a number different from 3, so $ABC^c = \emptyset$. Thus

$$P(A(B - C)) = P(ABC^c) = 0$$

which implies that $A$ and $B - C$ are not independent.

2. [Ghahramani, 3.5, # 11]

Let $A_i$ denote the event that bit $i$ is received without error. It is given that $P(A_i^c) = 0.0001$, so $P(A_i) = 0.9999$ for $i = 1, \ldots, 64$. It is also given that each bit is affected by errors independently of other bits; i.e., the events $A_i$ are independent. Hence, the probability that a message of 64 bits is received without error is

$$P\left(\bigcap_{i=1}^{64} A_i\right) = \prod_{i=1}^{64} P(A_i) = (0.9999)^{64} = 0.9936.$$ 

3. [Ghahramani, 3.5, # 22]

Let $B_i$ denote the event that a match occurs on the $i$th roll. Then the $B_i$, $i = 1, \ldots, 6$ are independent and $P(B_i) = 1/6$. Letting $B$ denote the event that at least one match occurs, we have $B^c = \bigcap_{i=1}^{6} B_i^c$, so

$$P(B) = 1 - P(B^c) = 1 - \left(\frac{5}{6}\right)^6 = 0.6651.$$
4. [Ghahramani, 3.5, # 29]

Let $A_i$, $i = 1, \ldots, 6$, denote the event that the $i$th switch is closed. Then there is a path from the input to the output if and only if either $A_1A_2$ or $A_3A_4$ (or both) occur, and in addition, $A_5$ occurs. Thus the event $B$ that the current flows can be expressed as

$$B = A_1(A_2A_4 \cup A_3A_5)A_6.$$

Since the $A_i$ are independent and have equal probability $p$,

$$P(B) = P(A_1(A_2A_4 \cup A_3A_5)A_6)$$

$$= P(A_1A_2A_4A_6 \cup A_1A_3A_5A_6)$$

$$= P(A_1A_2A_4A_6) + P(A_1A_3A_5A_6) - P(A_1A_2A_3A_4A_5A_6)$$


$$= 2p^4 - p^6$$

$$= p^4(2 - p^2).$$

5. [A monkey at a typewriter . . . ]

(a) The number of possible permutations of the 26 letters of the English alphabet is $26!$. To count the number of these permutations in which “HAMLET” appears as a word, we count the total number of ways in which “HAMLET” and the remaining 20 letters of the alphabet can be arranged. This is simply the total number of permutations of 21 objects, which is 21!.

So, the probability that the word “HAMLET” appears in a random permutation of the 26 letters of the English alphabet is $21!/26! = 1/7893600 \approx 1.267 \times 10^{-7}$.

(b) Note that by the result of part (a), the probability that the word “HAMLET” does not appear in a random permutation of the letters of the alphabet is $1 - 1/7893600 \approx 0.99999987$.

So, if we have $n$ independent monkeys each typing out a random permutation of the 26 letters of the alphabet, the probability that none of them produces the string “HAMLET” is $(1 - 1/7893600)^n$.

Therefore, the probability that at least one of them produces the string “HAMLET” is

$$1 - (1 - 1/7893600)^n.$$

We need to determine the values of $n$ that would satisfy $1 - (1 - 1/7893600)^n > 0.9$, or equivalently, $(1 - 1/7893600)^n < 0.1$. Taking base-10 logarithms, we obtain

$$n \log_{10}(1 - 1/7893600) < -1$$

and since $\log_{10}(1 - 1/7893600) \approx -5.50185 \times 10^{-8}$, we see that

$$n > \frac{-1}{-5.50185 \times 10^{-8}} \approx 18175685.$$
So, we need more than 18,175,685 monkeys typing independently at random to ensure that, with probability greater than 0.9, at least one of them types the word “HAMLET”.

6. [ For the sake of argument . . . ]

Let $E_2$ denote the event that a two-engine plane completes its journey successfully, and let $E_4$ denote the corresponding event for a four-engine plane. We have

$$P(E_2) = P(\text{at least one engine does not fail}) = 1 - P(\text{both engines fail}) = 1 - (0.4)^2 = 0.84$$

and

$$P(E_4) = P(\text{at least two engines do not fail}) = 1 - [P(\text{all four engines fail}) + P(\text{exactly three engines fail})] = 1 - ((0.4)^4 + 4 (0.4)^3 (0.6)) = 0.8208$$

Thus, perhaps surprisingly, the two-engine plane is more likely to complete its flight successfully than a four-engine one.

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**Bonus question** [Let $S = \{1, 2, \ldots, n\}$ . . .]

(a) Let $N(B)$ denote the number of elements in $B$. By the law of total probability

$$P(A \subset B) = \sum_{i=0}^{n} P(A \subset B|N(B) = i)P(N(B) = i).$$

Since there are $\binom{n}{i}$ subsets of $\{1, 2, \ldots, n\}$ of size $i$, and $2^n$ subsets altogether, we have

$$P(N(B) = i) = \frac{\binom{n}{i}}{2^n}.$$ 

On the other hand, since there are $2^i$ subsets of any given $B$ with $N(B) = i$ and since $A$ is equally likely to be any subset of $\{1, 2, \ldots, n\}$ *independently* of what $B$ is,

$$P(A \subset B|N(B) = i) = \frac{2^i}{2^n}.$$ 

Thus

$$P(A \subset B) = \sum_{i=0}^{n} 2^{i-n} \frac{\binom{n}{i}}{2^n} = \sum_{i=0}^{n} \binom{n}{i} 2^i 4^{-n}.$$
\[ \begin{align*}
&= 4^{-n} \sum_{i=0}^{n} \binom{n}{i} 2^{i} 1^{n-i} \\
&= 4^{-n}(1 + 2)^{n} \quad \text{(by the binomial theorem)} \\
&= \left(\frac{3}{4}\right)^{n}.
\end{align*} \]

(b) Using again the law of total probability

\[ P(AB = \emptyset) = \sum_{i=0}^{n} P(AB = \emptyset | N(B) = i) P(N(B) = i). \]

Clearly,

\[ P(AB = \emptyset | N(B) = i) = \frac{2^{n-i}}{2^{n}} \]

and so

\[ P(AB = \emptyset) = \sum_{i=0}^{n} 4^{-n} \binom{n}{i} 2^{n-i} = 4^{-n} \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} 1^{i} \]
\[ = 4^{-n}(1 + 2)^{n} = \left(\frac{3}{4}\right)^{n}. \]

Note that we could have obtained the answer directly from part (a) since \(AB = \emptyset\) if and only if \(A \subset B^{c}\), and \(A\) and \(B^{c}\) are again independently, equally likely to be any of the subsets of \(\{1, 2, \ldots, n\}\).