1. [Ghahramani, 4.2, # 2]

The set of possible values of $X$ is $\mathcal{X} = \{-6, -2, -1, 2, 3, 4\}$. The event that two chips of the same given color are drawn has probability $\binom{5}{2} \binom{15}{2}$, so that

$$P(X = -6) = P(X = 2) = P(X = 4) = \frac{\binom{5}{2}}{\binom{15}{2}} = 0.095.$$  

The probability that two chips of two given different colors are drawn is $\binom{5}{1} \binom{5}{1} \binom{15}{2}$, so we obtain

$$P(X = -2) = P(X = -1) = P(X = 3) = \frac{\binom{5}{1} \binom{5}{1}}{\binom{15}{2}} = 0.238.$$  

The distribution function of $X$ can be calculated from

$$F(x) = \sum_{x_i \in \mathcal{X}: x_i \leq x} P(X = x_i).$$

For notational convenience, define $p = \frac{\binom{5}{2}}{\binom{15}{2}}$ and $q = \frac{\binom{5}{1} \binom{5}{1}}{\binom{15}{2}}$. Then $F(x)$ is given by

$$F(x) = \begin{cases} 
0 & \text{if } x < -6 \\
p & \text{if } -6 \leq x < -2 \\
p + q & \text{if } -2 \leq x < -1 \\
p + 2q & \text{if } -1 \leq x < 2 \\
2p + 2q & \text{if } 2 \leq x < 3 \\
2p + 3q & \text{if } 3 \leq x < 4 \\
1 & \text{if } x \geq 4 
\end{cases}$$
2. [Ghahramani, 4.2, # 12]

We are given

\[ F(t) = \begin{cases} 
(1/2)e^t & \text{if } t < 0 \\
1 - (3/4)e^{-t} & \text{if } t \geq 0.
\end{cases} \]

We need to determine whether \( F \) is non-decreasing, right-continuous, with \( \lim_{t \to -\infty} F(t) = 0 \) and \( \lim_{t \to \infty} F(t) = 1 \).

![Figure 1: Graph of \( F(t) \).](image)

As can be seen from the above plot, \( F(t) \) is certainly not non-decreasing, since \( F(0) = 0.25 < F(0^-) \). Therefore, \( F \) is not a distribution function. Note, however, that \( F \) is right-continuous, and does satisfy \( \lim_{t \to -\infty} F(t) = 0 \) and \( \lim_{t \to \infty} F(t) = 1 \).

3. [Let \( X \) be a real number selected at random from the interval \([0,1] \ldots \)]

(a) To find \( P(\frac{X}{1+X} \leq 2/5) \), we must determine the range of values of \( X \) that satisfy \( \frac{X}{1+X} \leq 2/5 \). Re-arranging the inequality yields \( X \leq (2/5)(1 + X) \), which is the same as \( (1 - 2/5)X \leq 2/5 \), i.e., \( X \leq \frac{2/5}{1-2/5} = 2/3 \). Thus,

\[ P\left(\frac{X}{1+X} \leq 2/5\right) = P(X \leq 2/3) = P([0,2/3]) = 2/3. \]

(b) We are given \( Y = X/(1 + X) \) and asked to determine \( F_Y(t) = P(Y \leq t) \).

Note that \( X \in [0,1] \) implies that \( Y = \frac{X}{1+X} \in [0,1/2] \). Hence, for any \( t \leq 0 \), we must have \( P(Y \leq t) = 0 \), and for any \( t \geq 1/2 \), we must have \( P(Y \leq t) = 1 \). So, it only remains to determine \( F_Y(t) \) for \( t \in (0,1/2) \).
Now, for any \( t \in (0, 1/2) \), we have

\[
P(Y \leq t) = P(\frac{X}{1+X} \leq t)
\]

\[
= P(X \leq t(1 + X)) = P(X(1 - t) \leq t)
\]

\[
= P(X \leq \frac{t}{1-t}) = P([0, \frac{t}{1-t}]) = \frac{t}{1-t}.
\]

Thus, overall, we see that

\[
F_Y(t) = P(Y \leq t) = \begin{cases} 
0 & \text{if } t \leq 0 \\
\frac{t}{1-t} & \text{if } 0 < t < 1/2 \\
1 & \text{if } t \geq 1/2.
\end{cases}
\]

4. [Suppose an urn contains three balls...]

(a) Let \( B_n, W_n, \) and \( G_n \) denote the events that no black ball, respectively no white or green ball, appears in the first \( n \) draws. The we have that \( \{X > n\} = B_n \cup W_n \cup G_n \). Thus we can use the inclusion-exclusion formula for 3 events:

\[
P(X > n) = P(B_n \cup W_n \cup G_n)
\]

\[
= P(B_n) + P(W_n) + P(G_n) - P(B_nW_n) - P(B_nG_n) - P(W_nG_n) + P(B_nW_nG_n).
\]

By independence, we have for all \( n \geq 1 \)

\[
P(B_n) = P(W_n) = P(G_n) = \left(1 - \frac{1}{3}\right)^n = \left(\frac{2}{3}\right)^n 
\]

\[
P(B_nW_n) = P(B_nG_n) = P(W_nG_n) = \left(1 - \frac{2}{3}\right)^n = \left(\frac{1}{3}\right)^n
\]
and, clearly,

\[ P(B_n W_n G_n) = 0. \]

Thus for all \( n \geq 1, \)

\[ P(X > n) = 3 \left( \frac{2}{3} \right)^n - 3 \left( \frac{1}{3} \right)^n = \frac{2^n - 1}{3^{n-1}}. \]

(b) The set of possible values of \( X \) is \( X = \{3, 4, 5, \ldots\} \) since one needs at least 3 draws for all three colors to appear at least once. For \( n \in X \) we have

\[
P(X = n) = P(X > n - 1) - P(X > n) = \frac{2^{n-1} - 1}{3^{n-2}} - \frac{2^n - 1}{3^{n-1}} = \frac{2^{n-1} - 2}{3^{n-1}}.
\]

Thus the pmf of \( X \) is

\[
p(n) = \frac{2^{n-1} - 2}{3^{n-1}}, \quad n = 3, 4, \ldots
\]

5. [A gambling book…]

(a) We assume that red wins independently for each spin of the roulette wheel with probability \( \frac{18}{38} \). Thus, if \( R_i \) denotes the event that a bet of red wins on the \( i \)th spin of the wheel, then \( R_1, R_2, \) and \( R_3 \) are independent events. On each $1 bet on red, the gambler’s net winning is either $1 = $2 - $1, or $ -1 (i.e., a $1 loss). Thus following the book’s strategy results in

\[
X = \begin{cases} 
1 & \text{if } R_1 \text{ occurs} \\
1 & \text{if } R_1^c R_2 R_3 \text{ occurs} \\
-1 & \text{if } R_1^c R_2^c R_3 \text{ or } R_1^c R_2 R_3^c \text{ occur} \\
-3 & \text{if } R_1^c R_2^c R_3^c \text{ occurs.}
\end{cases}
\]

We know that \( P(R_i) = \frac{18}{38} \), and by independence

\[
P(R_1^c R_2 R_3) = \left( \frac{20}{38} \right)^2 \left( \frac{18}{38} \right), \quad P(R_1^c R_2^c R_3) = P(R_1^c R_2 R_3^c) = \left( \frac{20}{38} \right)^2 \frac{18}{38}, \quad P(R_1^c R_2^c R_3^c) = \left( \frac{20}{38} \right)^3.
\]

Therefore the pmf of \( X \) is

\[
P(X = x) = \begin{cases} 
\frac{18}{38} + \frac{20}{38} \left( \frac{18}{38} \right)^2 & \text{if } x = 1 \\
2 \left( \frac{20}{38} \right)^2 \frac{18}{38} & \text{if } x = -1 \\
\left( \frac{20}{38} \right)^3 & \text{if } x = -3.
\end{cases}
\]

From this we obtain

\[
P(X > 0) = P(X = 1) = \frac{18}{38} + \frac{20}{38} \left( \frac{18}{38} \right)^2 \approx 0.5918.
\]
(b) Using the definition of expectation

\[ E[X] = \sum_{x} xP(X = x) = 1 \cdot \left( \frac{18}{38} + \frac{20}{38} \left( \frac{18}{38} \right)^2 \right) - 1 \cdot \left( \frac{2}{38} \left( \frac{18}{38} \right)^2 \right) - \frac{3}{38} \left( \frac{20}{38} \right)^3 \approx -0.108. \]

Thus, on the average, the gambler loses $0.108 if she follows the strategy in the book.

6. [Ghahramani, 4.4, # 12]

From the linearity of the expectation


Since \( X \) is an integer picked at random from the set \( \{1, 2, \ldots, 10\} \), we have

\[ P(X = i) = \frac{1}{10}, \quad i = 1, \ldots 10. \]

Thus

\[ E[X] = \sum_{i=1}^{10} iP(X = i) = \frac{1}{10} \sum_{i=1}^{10} i = \frac{1}{10} \cdot \frac{10 \cdot 11}{2} = 5.5 \]

and

\[ E[X^2] = \sum_{i=1}^{10} i^2P(X = i) = \frac{1}{10} \sum_{i=1}^{10} i^2 = \frac{1}{10} \cdot \frac{10 \cdot 11 \cdot 21}{6} = 38.5 \]

and we obtain

\[ E[X(11 - X)] = 11 \cdot 5.5 - 38.5 = 22. \]

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**Bonus question**

(a) For \( k = 0, 1, 2, \ldots \), \( P(X > k) = \sum_{j=k+1}^{\infty} P(X = j) \). Therefore,

\[
\sum_{k=0}^{\infty} P(X > k) = \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} P(X = j) \\
\quad \overset{(\ast)}{=} \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} P(X = j) \\
\quad = \sum_{j=1}^{\infty} jP(X = j) \\
\quad = E[X].
\]

The equality \((\ast)\) above is simply a change in the order of summation. [To be absolutely rigorous, this can be done because all the terms in the sum are non-negative.]
(b) $X > k$ is the event that a red ball was drawn in each of the first $k$ draws. For $j = 1, 2, 3, \ldots$, let $R_j$ denote the event that a red ball is drawn on the $j$th draw. We are interested in the probability of the intersection

$$R_1 R_2 \cdots R_k.$$

Clearly, $P(R_1) = 1/2$. Now, $P(R_1 R_2) = P(R_1)P(R_2|R_1) = (1/2)P(R_2|R_1)$. Note that $P(R_2|R_1) = 2/3$, since, given that the first draw was red ($R_1$ occurred), there are 2 red balls and 1 black ball in the box at the time of the second draw. Therefore $P(R_1 R_2) = (1/2)(2/3) = 1/3$.

In general, a simple induction argument shows that for any $j \geq 2$, if the first $j - 1$ draws were all reds, then at the time of the $j$th draw, there are $j$ red balls and 1 black ball in the box, and hence,

$$P(R_j|R_1 R_2 \cdots R_{j-1}) = \frac{j}{j+1}.$$

Therefore, by the multiplication rule, for any $k \geq 1$,

$$P(R_1 R_2 \cdots R_k) = P(R_1) \times P(R_2|R_1) \times P(R_3|R_1 R_2) \times \cdots \times P(R_k|R_1 \cdots R_{k-1})$$

$$= \left(\frac{1}{2}\right) \left(\frac{2}{3}\right) \left(\frac{3}{4}\right) \cdots \left(\frac{k}{k+1}\right)$$

$$= \frac{1}{k+1}.$$

Hence, $P(X > k) = \frac{1}{k+1}$ for all $k \geq 1$. In fact, since $P(X > 0) = 1$, we have $P(X > k) = \frac{1}{k+1}$ for all $k \geq 0$.

(c) Since the random variable $X$ of part (b) takes values in the set $\{1, 2, 3, \ldots\}$, the result of part (a) can be applied to compute $E[X]$:

$$E[X] = \sum_{k=0}^{\infty} P(X > k) = \sum_{k=0}^{\infty} \frac{1}{k+1}.$$

Since the above sum diverges (i.e., is not finite), we have that $E[X] = \infty$. In the terminology we introduced in class, this means that $E[X]$ does not exist.