1. [Ghahramani, 4.5, # 9]

Using the definition on page 180 of the text, we calculate

\[ P(|X - 0| \leq t) = \begin{cases} 
0 & \text{if } t < 1 \\
1 & \text{if } t \geq 1 
\end{cases} \]

and

\[ P(|Y - 0| \leq t) = \begin{cases} 
0 & \text{if } t < 10 \\
1 & \text{if } t \geq 10. 
\end{cases} \]

Hence \( P(|Y - 0| \leq t) \leq P(|X - 0| \leq t) \), so by the Definition on p. 180, \( X \) is more concentrated about 0 than \( Y \).

However, we did not cover this definition in class so next we give another solution based on what we did cover. We argued in class that the variance of measures the spread of a random variable about its mean. We observe that \( E(X) = E(Y) = 0 \), and calculating \( \text{Var}(X) = E(X^2) = 1 \) and \( \text{Var}(Y) = E(Y^2) = 100 \), we conclude that \( X \) more concentrated about 0 than \( Y \) by this measure too.

2. [Ghahramani, 5.1, # 13]

Let’s count it a success when the price moves up 1/8th of a point and a failure when the price moves down 1/8th of a point. After six days the the stock has its original price exactly when there are an equal number (3) of successes and failures. Thus we need the probability of 3 successes in 6 independent Bernoulli trials with success probability \( p = 1/3 \). The desired probability is \( P(X = 3) \), where \( X \) is a binomial r.v. with parameters \((n, p) = (6, 1/3)\), which is

\[ P(X = 3) = \binom{6}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 \approx 0.219. \]

3. [Jack and Jill again play a series of chess games . . .]

Let \( E_4 \) be the event that the series ends in 4 games, and let \( J \) be the event that Jack wins the series, so that \( J^c \) is the event that Jill wins the series.

We want \( P(J \mid E_4) = \frac{P(JE_4)}{P(E_4)} \). Note that \( E_4 \) is the union of the mutually exclusive events \( JE_4 \) and \( J^cE_4 \), so that \( P(E_4) = P(JE_4) + P(J^cE_4) \). In summary,

\[ P(J \mid E_4) = \frac{P(JE_4)}{P(JE_4) + P(J^cE_4)}. \]
Now, $JE_4$ is the event that Jack wins the series in 4 games. This happens if and only if Jack wins game 4, as well as two of the first three games. The probability that Jack wins game 4 is 0.4, and the probability that he wins two of the first three games is $\binom{3}{2}(0.4)^2(0.6) = (\frac{3}{2})(0.4)^3(0.6)$.

Therefore, $P(JE_4) = (0.4)\binom{3}{2}(0.4)^2(0.6) = (\frac{3}{2})(0.4)^3(0.6)$.

A similar calculation shows that $P(J^cE_4) = (\frac{3}{2})(0.3)^3(0.7)$. We thus find that

$$P(J \mid E_4) = \frac{(\frac{3}{2})(0.4)^3(0.6)}{(\frac{3}{2})(0.4)^3(0.6) + (\frac{3}{2})(0.3)^3(0.7)} \approx 0.67.$$

4. [Ghahramani, 5.3, # 6]

We shall assume that each shot attempt is independent of any other attempt. Let $B$ denote the event “basket made,” and $N = B^c$ denote the event “basket not made.” So, $P(B) = 0.45$, $P(N) = 0.55$.

(a) $P(1st\ basket\ on\ 6th\ shot) = P(\text{NNNNNB}) = (0.55)^5(0.45) \approx 0.0226$.

(b) The event that the first and second baskets are made on the fourth and eighth shots, respectively, is simply $\text{NNNBNNNB}$. So, the probability of this occurring is $P(\text{NNNBNNNB}) = (0.55)^6(0.45)^2 \approx 0.0056$.

5. [Ghahramani, 5.3, # 14]

Let’s call the event that a senior citizen enters the mall a success. Since 15% of the population are senior citizens, the probability of success is $p = 0.15$. Assuming people enter the mall independently of each other, the number of people that enter the mall until the 10th senior citizen enters is the number of Bernoulli trials until a total of 10 successes accumulate. This number, $Y$, is a negative binomial random variable with parameters $(10, 0.15)$. Note that the random variable $X$ is given in terms of $Y$ as

$$X = Y - 10.$$

Recall that the pmf of $Y$ is

$$P(Y = n) = \left(\frac{n-1}{9}\right)(0.15)^{10}(0.85)^{n-10}, \quad n = 10, 11, 12, \ldots$$

Thus we obtain

$$P(X = i) = P(Y - 10 = i) = P(Y = i + 10) = \left(\frac{i+9}{9}\right)(0.15)^{10}(0.85)^i, \quad i = 1, 2, 3, \ldots$$

6. [Ghahramani, 5.2, # 16]

Let $N(t)$ be the number of customers arriving at the bookstore in a time period of duration $t$ hours. We are given that $N(t)$ is a Poisson process with rate $\lambda = 6$. 

2
The event “exactly one customer arrives between 9:30am and 10:00am, and exactly 10 customers arrive between 9:30am and noon”, can be re-worded as “exactly one customer arrives between 9:30am and 10:00am, and exactly 9 customers arrive between 10:00am and noon”. So, if $A$ is the event “exactly one customer arrives between 9:30am and 10:00am”, and $B$ is the event “exactly 9 customers arrive between 10:00am and noon”, we are looking for the probability $P(AB)$.

Note that the time interval from 9:30 to 10:00 am is disjoint from the time interval from 10:00am to noon, so events occurring within these two time intervals are independent. This is just the independent increments assumption that we used to derive the Poisson process.

Thus, $A$ and $B$ are independent events, so $P(AB) = P(A)P(B)$. Now,

$$P(A) = P(N(1/2) = 1) = \frac{e^{-(1/2)(6)}[(1/2)(6)]^1}{1!} = 3e^{-3}$$

and

$$P(B) = P(N(2) = 9) = \frac{e^{-(2)(6)}[(2)(6)]^9}{9!} = \frac{12^9e^{-12}}{9!}.$$

So, the required probability is

$$P(AB) = P(A)P(B) = (3e^{-3}) \left( \frac{12^9e^{-12}}{9!} \right) \approx 0.01305.$$

7. [Each day, a certain website . . .]

Let $N(t)$ denote the number of hacker attacks in a time period of $t$ consecutive days. We are given that $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda = 2$.

(a) The website remains online at the end of the day if and only if there have been 5 or fewer attacks that day. So, the required probability is

$$P(N(1) \leq 5) = \sum_{n=0}^{5} P(N(1) = n) = \sum_{n=0}^{5} \frac{e^{-(2)}[(2)(1)]^n}{n!} = \sum_{n=0}^{5} \frac{1}{n!} \left[ 1 + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} \right] = \frac{109}{15} e^{-2} \approx 0.9834.$$

(b) The number of consecutive days the website remains online is the number of days until it happens that there are more than 5 hacker attacks in one day. Define “success” to be event “more than 5 hacker attacks in one day”. Note that

$$P(\text{success}) = P(N(1) > 5) = 1 - P(N(1) \leq 5) \approx 1 - 0.9834 = 0.0166.$$
By the independent increments assumption (Assumption 3) used to derive the Poisson process, events happening on any particular day are independent of events happening on any other day. So, if we take each day to be a “trial”, each trial has two possible outcomes (success and failure), and the trials are independent of each other. The trials thus forms a sequence of independent Bernoulli trials.

Let \( X \) be the random variable that counts the number of trials until the first success occurs. Since the trials are independent, \( X \) is a geometric random variable with parameter \( p = P(\text{success}) = 0.017 \). Therefore, the expected number of consecutive days that the website remains online is

\[
E[X] = \frac{1}{p} \approx 60.24.
\]

**Bonus question [Ghahramani, 4.5, # 14]**

We will use the fact, known from calculus, that \( \sum_{n=1}^{\infty} \frac{1}{n^\alpha} < \infty \) if \( \alpha > 1 \) and \( \sum_{n=1}^{\infty} \frac{1}{n^\alpha} = \infty \) otherwise.

First we show that \( \sum_{n=1}^{\infty} n^3 p(n) \) is a convergent series. Since \( x_n = (-1)^n \sqrt{n} \) and \( p(n) = \frac{6}{(\pi n)^2} \), we have that

\[
\sum_{n=1}^{\infty} x_n^3 p(n) = \sum_{n=1}^{\infty} ((-1)^n \sqrt{n})^3 p(n) = \frac{6}{\pi^2} \sum_{n=1}^{\infty} (-1)^{3n} \frac{1}{\sqrt{n}}
\]

\[
= -\frac{6}{\pi^2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}}
\]

\[
= -\frac{6}{\pi^2} \sum_{k=1}^{\infty} \left( \frac{1}{\sqrt{2k-1}} - \frac{1}{\sqrt{2k}} \right).
\]

The \( k \)th term in the summation can be rewritten as

\[
\frac{1}{\sqrt{2k-1}} - \frac{1}{\sqrt{2k}} = \frac{\sqrt{2k} - \sqrt{2k-1}}{\sqrt{2k(2k-1)}}.
\]

Define \( f(x) = \sqrt{x} \), so that \( f'(x) = \frac{1}{2\sqrt{x}} \). Then by the mean value theorem of differentiation we have, for some \( y \in [2k-1, 2k] \),

\[
\sqrt{2k} - \sqrt{2k-1} = f(2k) - f(2k-1) = f'(y)(2k - (2k - 1)) = \frac{1}{2\sqrt{y}} \leq \frac{1}{2\sqrt{2k-1}}.
\]

Thus we get the upper bound

\[
\frac{1}{\sqrt{2k-1}} - \frac{1}{\sqrt{2k}} \leq \frac{\sqrt{2k} - 1}{\sqrt{2k(2k-1)}} \leq \frac{1}{2(2k-1)^{3/2}}.
\]
Since $\sum_{k=1}^{\infty} (2k - 1)^{3/2} < \infty$, we obtain that the series $\sum_{n=1}^{\infty} x_n^3 p(n)$ converges.

On the other hand,

$$\sum_{n=1}^{\infty} |x_n^3| p(n) = \sum_{n=1}^{\infty} ((\sqrt{n})^3 1/n^2 = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

and since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$, $\sum_{n=1}^{\infty} |x_n^3| p(n)$ does not converge, and thus $E(X^3)$ does not exist.