1. [Ghahramani, 6.1, # 4]

(a) We have to calculate the probability \( P(10000X < 15000) \). Since \( 10000X < 15000 \) if and only if \( X < 1.5 \), this can be done as follows:

\[
P(10000X < 15000) = P(X < 1.5) = \int_{-\infty}^{1.5} f(x) \, dx = \int_{1}^{3/2} \frac{2}{x^2} \, dx = \left[-\frac{2}{x}\right]_{1}^{3/2} = 1 - \frac{2}{3} = \frac{2}{3}.
\]

Thus approximately 66% of the tires will last fewer than 15,000 miles.

(b) Let \( A \) denote the event that a tire lasts between 10,000 and 12,500 miles, and let \( B \) be the event that it has a lifetime less than 15,000 miles. Then we have to calculate the probability \( P(A|B) \). Since \( AB = A \), we have

\[
P(A|B) = \frac{P(A)}{P(B)} = \frac{P(1 < X < 1.25)}{P(X < 1.5)} = \frac{\int_{1}^{1.25} \frac{2}{x^2} \, dx}{\int_{1}^{3/2} \frac{2}{x^2} \, dx} = \frac{2/5}{2/3} = \frac{3}{5}.
\]

Thus 60% of the tires that have lifetimes fewer than 15,000 miles will last between 10,000 and 15,000 miles.

2. [Suppose the weakly demand...]

We have to find \( c \) such that \( P(1000X > c) = 0.01 \); or equivalently, \( P(X > c/10000) = 0.01 \). We know that \( P(X > x) = 1 - F(x) \), where \( F \) is the distribution function of \( X \), so we first determine \( F(x) = \int_{-\infty}^{x} f(t) \, dt \).

Clearly, \( F(x) = 0 \) if \( x \leq 0 \). For \( x \in (0, 1) \) we have

\[
F(x) = \int_{0}^{x} 5(1-t)^4 \, dt = \left[-(1-t)^5\right]_{0}^{x} = 1 - (1-x)^5.
\]

We also have \( F(x) = 1 \) if \( x \geq 1 \). Thus

\[
F(x) = \begin{cases} 
0 & \text{if } x \leq 0, \\
1 - (1-x)^5 & \text{if } 0 < x < 1, \\
1 & \text{if } x \geq 1.
\end{cases}
\]
Thus $c$ must satisfy $1 - F(x) = c$, or $F(x) = 0.99$. Solving

$$1 - (1 - x)^5 = 0.99$$

gives $x = 1 - (0.01)^{1/5} \approx 0.602$. We obtain that at least 602 liters of regular gasoline should be stored at the beginning of the week.

3. Let $X$ be a random number selected from the interval $[-2, 3]$. 

As derived in class, the pdf of a number $X$ randomly selected from the interval $[-2, 3]$ is

$$f_X(x) = \begin{cases} 
1/5 & \text{if } -2 \leq x \leq 3 \\
0 & \text{otherwise.}
\end{cases}$$

We are asked to determine the pdf of $Y = |X|$. We cannot use the method of transformations to do so, since the function $h(x) = |x|$ is certainly not invertible for $x \in [-2, 3]$; so we use the method of distribution functions.

The distribution function of $Y$ is given by $F_Y(t) = P(Y \leq t)$. Note that since $X$ takes values only in the interval $[-2, 3]$ (the pdf of $X$ is zero outside that interval), $Y = |X|$ only takes values in the interval $[0, 3]$. Therefore, we immediately have that $P(Y \leq t) = 0$ for $t < 0$, and $P(Y \leq t) = 1$ for $t > 3$. So, we only have to determine $P(Y \leq t)$ for $t \in [0, 3]$.

Now, $P(Y \leq t) = P(|X| \leq t) = P(X \in [-t, t])$. Note that if $t > 2$, the left end of the interval $[-t, t]$ goes outside the interval $[-2, 3]$. Since $X$ cannot take values outside $[-2, 3]$, we find that for $t > 2$, $P(X \in [-t, t]) = P(X \in [-2, t])$. This situation does not arise when $0 < t \leq 2$. We therefore have that when $0 < t \leq 2$,

$$P(Y \leq t) = P(X \in [-t, t]) = \int_{-t}^{t} 1/5 \, dx = 2t/5,$$

and when $2 < t \leq 3$,

$$P(Y \leq t) = P(X \in [-2, t]) = \int_{-2}^{t} 1/5 \, dx = (t + 2)/5.$$

Putting it all together, we obtain

$$F_Y(t) = \begin{cases} 
0 & \text{if } t < 0 \\
2t/5 & \text{if } 0 \leq t \leq 2 \\
(t + 2)/5 & \text{if } 2 < t \leq 3 \\
1 & \text{if } t > 3.
\end{cases}$$
Therefore, the probability density function of $Y$ is

$$f_Y(t) = \frac{d}{dt} F_Y(t) = \begin{cases} \frac{2}{5} & \text{if } 0 \leq t \leq 2 \\ \frac{1}{5} & \text{if } 2 < t \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

4. [Let $X$ be a continuous random variable with density function . . . ]

$$f_X(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$Y = e^{X/3}$.

(a) The equation $y = e^{x/3}$ can be solved uniquely for $x$: $x = 3 \ln y$. Therefore, the function $h(x) = e^{x/3}$ is invertible, with inverse $g(y) = 3 \ln y$. The inverse function is differentiable, the derivative being $g'(y) = 3/y$. So, we can use the method of transformations to determine the pdf of $Y$. By that method, we have that

$$f_Y(y) = f_X(g(y)) |g'(y)| = |3/y| f_X(3 \ln y) = \begin{cases} |3/y| e^{-3 \ln y} & \text{if } 3 \ln y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $3 \ln y > 0$ if and only if $y > 1$, and that $e^{-3 \ln y} = (e^{-\ln y})^3 = (1/y)^3 = 1/y^3$. Therefore,

$$f_Y(y) = \begin{cases} 3/y^4 & \text{if } y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) (i) Using the definition of $E[Y]$, we have that

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_{1}^{\infty} y (3/y^4) dy = \int_{1}^{\infty} 3/y^3 dy = 3 \left[ -1/(2y^2) \right]_{y=1}^{\infty} = 3/2.$$

(ii) Using the fact that $E[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x) dx$, we obtain

$$E[Y] = E[e^{X/3}] = \int_{-\infty}^{\infty} e^{x/3} f_X(x) dx = \int_{0}^{\infty} e^{x/3} e^{-x} dx = \int_{0}^{\infty} e^{-2x/3} dx = [-3/2 e^{-2x/3}]_{x=0}^{\infty} = 3/2.$$

5. [A stick of total length . . . ]
Let $L$ denote the length of the piece that contains the point $1/3$. Then $L = h(X)$, where $h$ is the function

$$h(x) = \begin{cases} 
1 - x & \text{if } x < 1/3 \\
 x & \text{if } x \geq 1/3.
\end{cases}$$

Since the pdf of $X$ is $f(x) = 1$ if $x \in [0, 1]$ and $f(x) = 0$ otherwise, we have

$$E[L] = E[h(X)] = \int_0^1 h(x) f(x) \, dx$$

$$= \int_0^1 h(x) \, dx$$

$$= \int_0^{1/3} (1 - x) \, dx + \int_{1/3}^1 x \, dx$$

$$= \frac{1}{2} - \frac{1}{2} \left(1 - \frac{1}{3}\right)^2 + \frac{1}{2} - \frac{1}{2} \left(\frac{1}{3}\right)^2$$

$$= \frac{13}{18}.$$ 

6. [Suppose that if you are...]

Let $X$ denote the travel time and let $C$ be your cost. If you leave $t$ minutes before the appointment and $X = x$, then you cost is $C = h(x)$, where

$$h(x) = \begin{cases} 
3(t - x) & \text{if } x \leq t \\
5(x - t) & \text{if } x > t.
\end{cases}$$

Since $f(x) = 0$ for all $t < 0$, we obtain

$$E[C] = \int_0^\infty h(x) f(x) \, dx$$

$$= \int_0^t 3(t - x) f(x) \, dx + \int_t^\infty 5(x - t) f(x) \, dx$$

$$= 3t \int_0^t f(x) \, dx - 3 \int_0^t x f(x) \, dx + 5 \int_t^\infty x f(x) \, dx - 5t \int_t^\infty f(x) \, dx.$$ 

We can easily obtain the optimal value of $t$ by differentiating $E[C]$ with respect to $t$ and setting the derivative to zero. The derivative is

$$\frac{d}{dt} E[C] = 3 \int_0^t f(x) \, dx + 3tf(t) - 3tf(t) - 5tf(t) - 5 \int_t^\infty f(x) \, dx + 5tf(t)$$

$$= 3F(t) - 5(1 - F(t))$$

$$= 8F(t) - 5.$$
where we used the fact that if \( g(x) \) is an integrable function that is continuous at \( t \), then
\[
\frac{d}{dt} \int_t^\infty g(x) \, dx = -g(t).
\]
Since the derivative \( 8F(t) - 5 \) is a strictly increasing function, \( E[C] \) will indeed be minimized by the \( t \) at which the derivative is zero. Thus the optimal \( t \) must be such that
\[
F(t) = \frac{5}{8}.
\]
Since \( F(t) = \int_0^t (1/10)e^{-x/10} \, dx = 1 - e^{-t/10} \) for all \( t > 0 \), we obtain that the unique \( t \) minimizing \( E[C] \) must satisfy
\[
1 - e^{-t/10} = \frac{5}{8}
\]
i.e.,
\[
t = 10 \ln \frac{8}{3} \approx 9.80.
\]
Thus you must leave approximately 10 minutes before the appointment to minimize your cost.

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**Bonus question**  [Let \( X \) be a continuous random variable...]  

For simplicity, we assume that the pdf of \( X \) is continuous (this assumption simplifies the proof). Then we have
\[
E(|X - c|) = \int_{-\infty}^\infty |t - c| f(t) \, dt
\]
\[
= \int_{-\infty}^c (c-t) f(t) \, dt + \int_c^\infty (t-c) f(t) \, dt
\]
\[
= cP(X \leq c) - \int_{-\infty}^c t f(t) \, dt - cP(X > c) + \int_c^\infty t f(t) \, dt
\]
\[
= c(2F(c) - 1) + \int_{-\infty}^c t f(t) \, dt + \int_c^\infty t f(t) \, dt
\]
where \( F(t) \) is the distribution function of \( X \). Note that the above implies that \( \lim_{c \to -\infty} E(|X - c|) = \lim_{c \to \infty} E(|X - c|) = \infty \). Since the expression for \( E(|X - c|) \) is differentiable, we can find its minimum by setting its derivative with respect to \( c \) to zero. We have
\[
\frac{d}{dc} E(|X - c|) = 2F(c) - 1 + 2cf(c) - cf(c) - cf(c)
\]
\[
= 2F(c) - 1.
\]
Setting the derivative to zero we obtain that the \( c \) minimizing \( E(|X - c|) \) must satisfy
\[
F(c) = P(X \leq c) = \frac{1}{2}.
\]
This $c$ is called the median of $X$.

The median of $X$ is in general not unique. For example, if $f(x)$ is given by

$$f(x) = \begin{cases} 
\frac{1}{2} & \text{if } 0 \leq x \leq 1, \\
\frac{1}{2} & \text{if } 2 \leq x \leq 3, \\
0 & \text{otherwise}, 
\end{cases}$$

then any $c \in [1, 2]$ satisfies $F(c) = \frac{1}{2}$. 