1. [Consider the function . . .]

We have

\[ f(x) = \begin{cases} 
  k(2x - x^2) & \text{if } 0 < x < 3/2 \\
  0 & \text{otherwise.}
\end{cases} \]

\( f(x) \) can be a probability density function (pdf) only if it is non-negative for all values of \( x \). Since \( 2x - x^2 = x(2-x) \) is positive throughout \( 0 < x < 2 \), it is certainly positive in the region \( 0 < x < 3/2 \).

So, any choice of \( k \geq 0 \) would make \( f(x) \geq 0 \) for all \( x \).

To make \( f(x) \) a pdf, \( k \) would have to be chosen so that \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \). Thus, we must have

\[
1 = \int_{0}^{3/2} k(2x - x^2) \, dx = k \left[ x^2 - \frac{x^3}{3} \right]_{x=0}^{x=3/2} = k \left( \frac{9}{8} \right).
\]

Hence, \( k = \frac{8}{9} \) would make \( f(x) \) a pdf.

2. [A random variable \( X \) has a pdf given by . . .]

\[ f_X(x) = \begin{cases} 
  \frac{2}{x^3} & \text{if } x > 1 \\
  0 & \text{otherwise.}
\end{cases} \]

(a) \( F_X(t) = P(X \leq t) = \int_{-\infty}^{t} f_X(x) \, dx. \)

For \( t \leq 1 \), since \( f_X(x) = 0 \) for all \( x \) in the range \(( -\infty, t ] \), we have \( F_X(t) = 0 \).

For \( t > 1 \), we find that

\[
F_X(t) = \int_{-\infty}^{t} f_X(x) \, dx = \int_{1}^{t} \frac{2}{x^3} \, dx = \left[ -\frac{1}{x^2} \right]_{x=1}^{x=t} = 1 - \frac{1}{t^2}.
\]

Thus,

\[
F_X(t) = \begin{cases} 
  0 & \text{if } t \leq 1 \\
  1 - \frac{1}{t^2} & \text{if } t \geq 1.
\end{cases}
\]

(b) By definition of conditional probability,

\[
P(X \geq 2 \mid X < 3) = \frac{P(X \geq 2 \text{ AND } X < 3)}{P(X < 3)} = \frac{P(2 \leq X < 3)}{P(X < 3)} = \frac{F_X(3) - F_X(2)}{F_X(3)}.
\]

It is important to note that in the last equality above, we have used the facts that \( P(X < 3) = P(X \leq 3) = F_X(3) \), and \( P(2 \leq X < 3) = P(2 < X \leq 3) = F_X(3) - F_X(2) \), both of which are true only because \( X \) is a continuous random variable.
Therefore, using the result of part (a), we have that

\[
P(X \geq 2 \mid X < 3) = \frac{(1 - 1/9) - (1 - 1/4)}{1 - 1/9} = \frac{5/36}{8/9} = \frac{5}{32}.
\]

(c) The mean of \( X \) is determined as follows:

\[
E[X] = \int_{-\infty}^{\infty} x f_X(x) \, dx = \int_{1}^{\infty} x \left(\frac{2}{x^3}\right) \, dx = 2 \left(\frac{\ln x}{x} \bigg|_{x=1}\right) = 2.
\]

To determine the variance, as usual, we use the formula \( \text{Var}(X) = E[X^2] - (E[X])^2 \). However, note that

\[
E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) \, dx = \int_{1}^{\infty} x^2 \left(\frac{2}{x^3}\right) \, dx = 2 \left(\ln x \bigg|_{x=1}\right) = \infty.
\]

Thus, \( E[X^2] \) does not exist, and therefore, \( \text{Var}(X) \) does not exist either.

3. [Ghahramani, 6.2, # 4]

We have that \( f(x) > 0 \) if \( x > 0 \) and \( f(x) = 0 \) if \( x \leq 0 \). Thus the set of possible values of \( X \) is \((0, \infty)\). The function \( h(x) = \log_2(x) \) is strictly increasing and maps \((0, \infty)\) onto the entire real line \( \mathbb{R} \). The inverse of \( h \) is \( g(y) = 2^y \) for \( y \in \mathbb{R} \). Thus we have \( g'(y) = (\ln 2)2^y \). Thus the formula

\[
f_Y(y) = f_X(g(y))|g'(y)|
\]

provides

\[
f_Y(y) = (3 \ln 2)2^y e^{-3(2^y)}
\]

for all \( y \in \mathbb{R} \).

4. [Let \( X \) be a random variable. . .]

Let \( h(x) = -x \) if \( x < 0 \) and \( h(x) = x^3 \) if \( x \geq 0 \). Then \( h(x) \) is strictly decreasing for \( x < 0 \) and strictly increasing for \( x \geq 0 \). Thus we cannot apply the method of transformations directly, so we’ll use the method of distribution functions.

Clearly, \( h(x) > 0 \) for all \( x \neq 0 \), so \( F_Y(y) = P(h(X) \leq y) = 0 \) if \( y \leq 0 \). For \( y > 0 \) we have that \( h(x) \leq y \) if and only if \( -y \leq x \leq y^{1/3} \) (sketching the graph of \( h(x) \) helps here). Thus

\[
F_Y(y) = P(h(X) \leq y) = P(-y \leq X \leq y^{1/3}) = F_X(y^{1/3}) - F_X(-y) \quad \text{for all } y > 0.
\]

Then, since \( f_X(x) = \frac{1}{2}e^{-|x|} \) for all \( x \), we obtain

\[
f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \left( F_X(y^{1/3}) - F_X(-y) \right)
\]

\[
= f_X(y^{1/3}) \left( \frac{1}{3} y^{-2/3} \right) - f_X(-y)(-1)
\]

\[
= \frac{1}{6} e^{-|y|^{1/3}} y^{-2/3} + \frac{1}{2} e^{-|y|}.
\]
Thus $f_X(y)$ is given by

$$f_Y(y) = \begin{cases} 
\frac{1}{6}e^{-|y|^{1/3}}y^{-2/3} + \frac{1}{2}e^{-|y|} & \text{if } y > 0 \\
0 & \text{otherwise.}
\end{cases}$$

5. [A vendor at a market . . .]

As mentioned in the hint, the key to this problem is that the vendor cannot sell more than the $C$ pounds of mushrooms he has in stock. Thus, if the demand, $X$, exceeds his supply, $C$, then he sells all $C$ pounds of his mushrooms, but no more, giving him a profit of $C$ dollars. On the other hand, if $X \leq C$, then the profit he makes is $4X - 3C$ dollars. So, if we let $Y$ denote the vendor’s daily profit, then we have

$$Y = \begin{cases} 
4X - 3C & \text{if } X \leq C \\
C & \text{if } X > C
\end{cases}$$

Thus, $Y = h(X)$, where $h$ is the function defined by

$$h(x) = \begin{cases} 
4x - 3C & \text{if } x \leq C \\
C & \text{if } x > C
\end{cases}$$

(a) The vendor’s expected profit is

$$E[Y] = E[h(X)] = \int_{-\infty}^{\infty} h(x) f(x) \, dx = \frac{1}{40} \int_{0}^{40} h(x) \, dx$$

$$= \frac{1}{40} \left( \int_{0}^{C} (4x - 3C) \, dx + \int_{C}^{40} C \, dx \right)$$

$$= \frac{1}{40} \left( -C^2 + C(40 - C) \right)$$

$$= C - \frac{C^2}{20}.$$ 

(b) To maximize the function $g(C) = C - C^2/20$, we set $g'(C) = 0$ which yields $C = 10$ as the only solution. Since $g''(10) = -1/10 < 0$, we conclude that $g(C)$ reaches its maximum at $C = 10$.

6. [Ghahramani, 6.3, # 9]

Let $X$ be the random variable representing the length of the side whose probability density function is given by

$$f_X(x) = \begin{cases} 
x/6 & \text{if } 2 < x < 4 \\
0 & \text{otherwise.}
\end{cases}$$
Since the hypotenuse has length 9, the remaining side of the right triangle has length given by
\[ L = \sqrt{81 - x^2}. \]
The expected value of \( L \) is
\[ E[L] = E[\sqrt{81 - X^2}] = \int_{-\infty}^{\infty} \sqrt{81 - x^2} f_X(x) \, dx = \int_{2}^{4} \sqrt{81 - x^2} \frac{x}{6} \, dx \]
\[ = \frac{1}{12} \int_{65}^{77} \sqrt{t} \, dt = \frac{1}{18} \left[ \frac{t^{3/2}}{3/2} \right]_{65}^{77} \approx 8.4236, \]
where we evaluated the integral involved using the change of variable \( t = 81 - x^2 \).

**Bonus question**

We are given a continuous random variable \( X \) with distribution function \( F(x) \), and are asked to determine the density function for \( Y = F(X) \). Since a distribution function maps \( \mathbb{R} \) to \([0,1]\), \( Y \) can only assume values in \([0,1]\). So, it is immediately clear that \( F_Y(t) = P(Y \leq t) = 0 \) for all \( t < 0 \), and \( F_Y(t) = 1 \) for all \( t \geq 1 \). It only remains to determine \( F_Y(t) \) for \( t \in [0,1] \).

We will show first that \( F_Y(t) = t \) for all \( t \in (0,1) \). This is relatively easy to see if \( F \) is a continuous strictly increasing function, so that it has an inverse \( F^{-1} : (0,1) \to \mathbb{R} \). For in this case, we have
\[ F_Y(t) = P(Y \leq t) = P(F(X) \leq t) = P(X \leq F^{-1}(t)) = F(F^{-1}(t)) = t, \]
where the penultimate equality simply uses the definition of the distribution function of \( X \).

However, it is not necessary to assume that \( F \) is invertible, as we now show. Fix an arbitrary \( t \in (0,1) \). Since \( F \) is the distribution function of a continuous r.v., it is a non-decreasing continuous function with \( \lim_{x \to -\infty} F(x) = 0 \) and \( \lim_{x \to \infty} F(x) = 1 \). It then follows by the intermediate value theorem from calculus that there exists an \( x \in \mathbb{R} \) such that \( F(x) = t \), and in fact, there must be a largest such \( x \in \mathbb{R} \). Let \( x_0 \) be the largest \( x \in \mathbb{R} \) such that \( F(x) = t \). The non-decreasing property of \( F \) ensures that \( F(x) \leq t \) if and only if \( x \leq x_0 \).

We are now ready to proceed. We have
\[ F_Y(t) = P(Y \leq t) = P(F(X) \leq t) = P(X \leq x_0) = F(x_0) = t, \]
with the last equality being a consequence of the fact that \( x_0 \) is by definition a point that satisfies \( F(x_0) = t \).

Therefore, we have shown that \( F_Y(t) = t \) for any \( t \in (0,1) \). By right-continuity of \( F_Y \), we also must have \( F_Y(0) = \lim_{t \to 0^+} F_Y(t) = \lim_{t \to 0^+} t = 0 \).
Putting all the pieces together, we see that

\[ F_Y(t) = \begin{cases} 
0 & t < 0 \\
t & 0 \leq t \leq 1 \\
1 & t > 1.
\end{cases} \]

Therefore, the probability density function of \( Y \) is

\[ f_Y(t) = \frac{d}{dt} F_Y(t) = \begin{cases} 
1 & 0 \leq t \leq 1 \\
0 & \text{otherwise}.
\end{cases} \]

Thus, \( Y \) is uniformly distributed over \([0, 1]\).