1. [Ghahramani, 8.1, # 4]

The pmf of $X$ and $Y$ is

$$p(x, y) = \begin{cases} \frac{1}{25}(x^2 + y^2) & \text{if } x = 1, 2, 3, \ y = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) To find $P(X > Y)$ we use that

$$P((X, Y) \in A) = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}: (x, y) \in A} p(x, y)$$

Thus

$$P(X > Y) = \sum_{(x, y): x > y} p(x, y) = p(1, 0) + p(2, 0) + p(2, 1)$$

$$= \frac{1}{25}(1^2 + 2^2 + 2^2 + 1^2)$$

$$= \frac{10}{25} = \frac{2}{5}$$

(b) Similarly

$$P(X + Y \leq 2) = \sum_{(x, y): x + y \leq 2} p(x, y)$$

$$= p(1, 0) + p(1, 1) + p(2, 0)$$

$$= \frac{1}{25}(1 + 1 + 2^2)$$

$$= \frac{7}{25}$$

(c) Finally,

$$P(X + Y = 2) = \sum_{(x, y): x + y = 2} p(x, y)$$

$$= p(1, 1) + p(2, 0)$$

$$= \frac{1}{25}(1 + 1 + 2^2)$$

$$= \frac{6}{25}$$
2. [Ghahramani, 8.1, # 11]

The pdf is given by

\[
f(x, y) = \begin{cases} 
\frac{1}{2} ye^{-x} & \text{if } x > 0, \ 0 < y < 2 \\
0 & \text{otherwise.}
\end{cases}
\]

Then we have for all \(x > 0\)

\[
f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy = \int_{0}^{2} \frac{1}{2} ye^{-x} \, dy
\]

\[
= \left[ \frac{1}{4} y^2 e^{-x} \right]_0^2 = e^{-x}.
\]

For \(x < 0\), we have \(\int_{-\infty}^{\infty} f(x, y) \, dy = 0\). Thus

\[
f_X(x) = \begin{cases} 
 e^{-x} & \text{if } x > 0 \\
0 & \text{otherwise}
\end{cases}
\]

i.e., \(X\) is an exponential random variable with parameter \(\lambda = 1\).

Similarly, for all \(0 < y < 2\),

\[
f_Y(y) = \int_{-\infty}^{\infty} f(x, y) \, dx = \int_{0}^{\infty} \frac{1}{2} ye^{-x} \, dx
\]

\[
= \left[ -\frac{1}{2} ye^{-x} \right]_0^{\infty} = \frac{1}{2} y
\]

and \(f_Y(y) = 0 \text{ for } y \notin (0, 2)\). Thus

\[
f_Y(y) = \begin{cases} 
 \frac{1}{2} y & \text{if } 0 < y < 2 \\
0 & \text{otherwise.}
\end{cases}
\]

3. [Ghahramani, 8.1, # 13]

The region \(R = \{(x, y) : x^2 \leq y \leq x\}\) in the problem is the shaded region in the above figure.
(a) \((X, Y)\) is uniformly distributed over the region \(R\), which means that the joint pdf of \(X\) and \(Y\) is given by

\[
f(x, y) = \begin{cases} 
\frac{1}{\text{area}(R)} & \text{if } (x, y) \in R \\
0 & \text{otherwise.}
\end{cases}
\]

The area of \(R\) is obtained by evaluating the double integral

\[
\text{area}(R) = \int \int_R \, dx \, dy = \int_0^1 \int_{x^2}^x \, dy \, dx = \int_0^1 (x - x^2) \, dx = \frac{1}{6}.
\]

Hence, the joint pdf of \(X\) and \(Y\) is

\[
f(x, y) = \begin{cases} 
6 & \text{if } x^2 \leq y \leq x \\
0 & \text{otherwise.}
\end{cases}
\]

(b) The marginal pdf of \(X\) is determined via

\[
f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy.
\]

Now, note that when \(x \notin [0, 1]\), \(f(x, y) = 0\) regardless of the value that \(y\) takes. Therefore, the above integral evaluates to 0 for all \(x \notin [0, 1]\).

For \(x \in [0, 1]\), we have

\[
f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy = \int_{x^2}^x 6 \, dy = 6(x - x^2).
\]

We have thus found that

\[
f_X(x) = \begin{cases} 
6(x - x^2) & \text{if } 0 \leq x \leq 1 \\
0 & \text{otherwise.}
\end{cases}
\]

The marginal pdf of \(Y\) is computed via

\[
f_Y(y) = \int_{-\infty}^{\infty} f(x, y) \, dx.
\]

Again, for \(y \notin [0, 1]\), \(f(x, y) = 0\) no matter the value of \(x\). Hence, the above integral evaluates to 0 for all \(y \notin [0, 1]\).

For \(y \in [0, 1]\), we have

\[
f_Y(y) = \int_{-\infty}^{\infty} f(x, y) \, dx = \int_{\sqrt{y}}^{\sqrt{y}} 6 \, dx = 6(\sqrt{y} - y).
\]

To summarize,

\[
f_Y(y) = \begin{cases} 
6(\sqrt{y} - y) & \text{if } 0 \leq y \leq 1 \\
0 & \text{otherwise.}
\end{cases}
\]
(c) \[ E[X] = \int_{-\infty}^{\infty} x f_X(x) \, dx = \int_{0}^{1} x (6(x - x^2)) \, dx = \int_{0}^{1} 6(x^2 - x^3) \, dx = 1/2. \]

\[ E[Y] = \int_{-\infty}^{\infty} y f_Y(y) \, dy = \int_{0}^{1} y (6(\sqrt{y} - y)) \, dy = \int_{0}^{1} 6(y^{3/2} - y^2) \, dy = 2/5. \]

We could also have calculated \( E[X] \) and \( E[Y] \) as follows:

\[ E[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y) \, dx \, dy = \int_{0}^{1} \int_{x^2}^{1} 6x \, dy \, dx = 1/2. \]

\[ E[Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x, y) \, dx \, dy = \int_{0}^{1} \int_{x^2}^{1} 6y \, dy \, dx = 2/5. \]

4. [The joint probability density function of \( X \) and \( Y \) is . . .]

We are given the joint pdf of \( X \) and \( Y \) as follows:

\[ f(x, y) = \begin{cases} 
24 xy & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1, \text{ and } 0 \leq x + y \leq 1, \\
0 & \text{otherwise}.
\end{cases} \]

(a) The marginal pdf of \( X \) is determined via

\[ f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy. \]

Now, note that when \( x \not\in [0, 1] \), \( f(x, y) = 0 \) regardless of the value that \( y \) takes. Therefore, the above integral evaluates to 0 for all \( x \not\in [0, 1] \).

For \( x \in [0, 1] \), we have

\[ f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy = \int_{x^2}^{1} 24xy \, dy = 12x(1 - x)^2 \]

We have thus found that

\[ f_X(x) = \begin{cases} 
12x(1 - x)^2 & \text{if } 0 \leq x \leq 1 \\
0 & \text{otherwise}.
\end{cases} \]

By symmetry, the marginal pdf of \( Y \) is given by

\[ f_Y(y) = \begin{cases} 
12y(1 - y)^2 & \text{if } 0 \leq y \leq 1 \\
0 & \text{otherwise}.
\end{cases} \]

(b) Since \( f(x, y) \neq f_X(x)f_Y(y) \), we find that the rv’s \( X \) and \( Y \) are not independent.
(c) The expected values in question can be computed as follows:

\[ E[X] = \int_{-\infty}^{\infty} x f_X(x) \, dx = \int_{0}^{1} x (12x(1-x)^2) \, dx = \int_{0}^{1} 12x^2(1-x)^2 \, dx = \frac{2}{5}. \]

\[ E[Y] = \int_{-\infty}^{\infty} y f_Y(y) \, dy = \int_{0}^{1} y (12y(1-y)^2) \, dy = \int_{0}^{1} 12y^2(1-y)^2 \, dy = \frac{2}{5}. \]

\[ E[X + Y] = E[X] + E[Y] = \frac{4}{5}. \]

\[ E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x,y) \, dx \, dy \]
\[ = \int_{0}^{1} \int_{0}^{1-y} xy (24xy) \, dx \, dy = \int_{0}^{1} \int_{0}^{1-y} 24x^2y^2 \, dx \, dy = \frac{2}{15}. \]

5. [Suppose two cars...]

The distance between \( X \) and \( Y \) is \(|X - Y|\), so we are asked to find the pdf of \( D = |X - Y| \). First we calculate the distribution function. Note that \( P(D < 0) = P(D > L) = 0 \), so we only have to consider \( P(D \leq t) \) for \( 0 \leq t \leq L \).

Since \( f(x,y) \) is the uniform pdf over the square \([0, L]^2\), the probability \( P(|X - Y| \leq t) \) is a geometric probability given by, for all \( 0 \leq t \leq L \),

\[ P(|X - Y| \leq t) = \frac{\text{area}([0, L]^2 \cap A_t)}{\text{area}([0, L]^2)}, \]

where

\[ A_t = \{(x, y) : |x - y| \leq t\}. \]

Note that since \(|x - y| \leq t\) is equivalent to \( x - t \leq y \leq x + t \), \( A_t \) is the diagonal strip bounded by the line \( y = x - t \) from below and by \( y = x + t \) from above. A sketch of the situation shows that the area of \([0, L]^2 \cap A_t\) is the area of \([0, L]^2\) minus the area of two disjoint isosceles right triangles of common side length \( L - t \). Thus

\[ F_D(t) = P(|X - Y| \leq t) = \frac{\text{area}([0, L]^2 \cap A_t)}{\text{area}([0, L]^2)} = \frac{L^2 - 2 \cdot \frac{(L-t)^2}{2}}{L^2} = 1 - \left(1 - \frac{t}{L}\right)^2 \]

and we obtain

\[ f_D(t) = \frac{d}{dt} F_D(t) = \frac{2}{L} \left(1 - \frac{t}{L}\right). \]

Thus

\[ f_D(t) = \begin{cases} \frac{2}{L} \left(1 - \frac{t}{L}\right) & \text{if } 0 \leq t \leq L \\ 0 & \text{otherwise.} \end{cases} \]
6. [Let \( X \) and \( Y \) be independent random variables. . .]

Since \( X \) and \( Y \) are independent rv’s, each of which is uniformly distributed over the interval \((0, 1)\), their joint pdf is given by

\[
f(x, y) = f_X(x)f_Y(y) = \begin{cases} 
1 & \text{if } 0 < x, y < 1 \\
0 & \text{otherwise.}
\end{cases}
\]

Denoting the unit square \( \{(x, y) : 0 \leq x, y \leq 1\} \) by \( \square \), we observe that since \((X, Y)\) is uniformly distributed over \( \square \), for any region \( R \) of the \((x, y)\)-plane,

\[
P((X, Y) \in R) = \frac{\text{area}(R \cap \square)}{\text{area}(\square)} = \text{area}(R \cap \square),
\]

since \( \text{area}(\square) = 1 \).

The pdf of \( Z = X/Y \) is determined by first finding the distribution function \( F_Z(t) = P(Z \leq t) \), and then differentiating to obtain \( f_Z(t) \).

Since \( X \) and \( Y \) are both in \((0, 1)\), \( Z = X/Y \) must be positive, and so \( F_Z(t) = P(Z \leq t) = 0 \) for \( t \leq 0 \).

So, let us now consider the case of \( t > 0 \). Here, we need

\[
F_Z(t) = P(Z \leq t) = P(X/Y \leq t) = P((X, Y) \in R) = \text{area}(R \cap \square),
\]

where \( R \) is the region \( R = \{(x, y) : y \geq x/t\} \). The region \( R \cap \square \) is shown in the figures below; note that the shape of the region depends on whether \( 0 < t < 1 \) or \( t \geq 1 \).

![Region](image)

Note that the area of the shaded region in the figures is \( \frac{1}{2}t \) when \( 0 < t < 1 \), and is \( 1 - \frac{1}{2}(1/t) \) when \( t \geq 1 \). Therefore,

\[
F_Z(t) = \begin{cases} 
0 & \text{if } t \leq 0 \\
\frac{t}{2} & \text{if } 0 < t < 1 \\
1 - \frac{1}{2t} & \text{if } t \geq 1
\end{cases}
\]
Finally, differentiating $F_Z(t)$, we obtain

$$f_Z(t) = \frac{d}{dt} F_Z(t) = \begin{cases} 
0 & \text{if } t \leq 0 \\
\frac{1}{2} & \text{if } 0 < t < 1 \\
\frac{1}{2t^2} & \text{if } t \geq 1.
\end{cases}$$

**Bonus question** [Ghahramani, 8.2, # 20]

We are given the function

$$f(x, y) = g(x)h(y)(1 + \alpha[2G(x) - 1][2H(y) - 1]),$$

for some $\alpha \in [-1, 1]$, where $g$ and $h$ are two probability density functions, with corresponding distribution functions $G$ and $H$, respectively.

To verify that $f(x, y)$ is a valid joint pdf, we need to show that $f(x, y) \geq 0$ for all $x, y \in \mathbb{R}$, and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1$.

To show that $f(x, y) \geq 0$, it is enough to show that $1 + \alpha[2G(x) - 1][2H(y) - 1] \geq 0$, since we know that $g(x), h(y) \geq 0$. We also know that $0 \leq F(x), G(x) \leq 1$, so that

$$-1 \leq 2G(x) - 1 \leq 1 \quad \text{and} \quad -1 \leq 2H(y) - 1 \leq 1.$$

Since we are given that $-1 \leq \alpha \leq 1$, it follows that

$$-1 \leq \alpha[2G(x) - 1][2H(y) - 1] \leq 1$$

as well. Thus, in particular, $\alpha[2G(x) - 1][2H(y) - 1] \geq -1$, or equivalently, $1 + \alpha[2G(x) - 1][2H(y) - 1] \geq 0$, as desired.

We next show that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1$. We have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y)(1 + \alpha[2G(x) - 1][2H(y) - 1]) \, dx \, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y) \, dx \, dy + \alpha \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y)[2G(x) - 1][2H(y) - 1] \, dx \, dy$$

(1)

Now,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y) \, dx \, dy = \left(\int_{-\infty}^{\infty} g(x) \, dx\right)\left(\int_{-\infty}^{\infty} h(y) \, dy\right) = (1)(1) = 1$$

(2)
since \( g(x) \) and \( h(x) \) are both pdf’s. On the other hand, since \( G(x) \) and \( H(x) \) are distribution functions and \( G'(x) = g(x) \), \( H'(x) = h(x) \), we see that

\[
\alpha \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y)[2G(x) - 1][2H(y) - 1] \, dx \, dy
\]

\[
= \alpha \left( \int_{-\infty}^{\infty} g(x)[2G(x) - 1] \, dx \right) \left( \int_{-\infty}^{\infty} h(y)[2H(y) - 1] \, dy \right)
\]

\[
= \alpha \left( [G^2(x) - G(x)]_{-\infty}^{\infty} \right) \left( [H^2(y) - H(y)]_{-\infty}^{\infty} \right)
\]

\[
= \alpha(0)(0) = 0
\]

The second equality above uses the fact that

\[
\frac{d}{dx}(G^2(x) - G(x)) = 2G(x)G'(x) - G'(x) = 2G(x)g(x) - g(x) = g(x)[2G(x) - 1],
\]

and similarly, \( \frac{d}{dy}(H^2(y) - H(y)) = h(y)[2H(y) - 1] \).

Finally, putting equations (2) and (3) into equation (1), we get

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1.
\]

Thus, we have shown that \( f(x, y) \) is the joint pdf of some random variables \( X \) and \( Y \). Now for the marginals. To obtain the marginal pdf of \( X \), we proceed as follows:

\[
f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy
\]

\[
= \int_{-\infty}^{\infty} g(x)h(y)(1 + \alpha[2G(x) - 1][2H(y) - 1]) \, dy
\]

\[
= \int_{-\infty}^{\infty} g(x)h(y) \, dy + \int_{-\infty}^{\infty} \alpha g(x)h(y)[2G(x) - 1][2H(y) - 1] \, dy
\]

\[
= g(x) \int_{-\infty}^{\infty} h(y) \, dy + \alpha g(x)[2G(x) - 1] \int_{-\infty}^{\infty} h(y)[2H(y) - 1] \, dy
\]

\[
= (g(x))(1) + \alpha g(x)[2G(x) - 1] \left( [H^2(y) - H(y)]_{-\infty}^{\infty} \right)
\]

\[
= g(x) + (\alpha)(0) = g(x).
\]

A similar argument shows that \( f_Y(y) = \int_{-\infty}^{\infty} f(x, y) \, dx = h(y) \).