Queen’s University
Department of Mathematics and Statistics

MTHE/STAT 351
Midterm Examination October 22, 2014

• Total points = 50. Duration = 2 hours.
• Closed book, closed notes.
• Simple calculators are permitted.
• Write the answers in the space provided, continue on the backs of pages if needed.
• SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks.
• Marks for each part of a question are shown in brackets at the right margin.

Marks: Please do not write in the space below.

Problem 1 [10]

Problem 2 [10]

Problem 3 [10]

Problem 4 [10]

Problem 5 [10]

Total: [50]
1. (a) Let $A$ and $B$ be events defined on some sample space, with $P(A) = 1/4$, $P(B) = 1/4$, and $P(A|B^c) = 1/6$. Find $P(A|B)$. 

\[ P(A|B) = \frac{P(AB)}{P(B)} \]

\[ AB \cup A B^c = A \quad \text{(disjoint union)} \]

\[ 5. \quad P(A) = P(AB) + P(A B^c) \Rightarrow P(AB) = P(A) - P(A B^c) \]

\[ P(A B^c) = P(A | B^c) P(B^c) = \frac{1}{4} \left( 1 - \frac{1}{4} \right) = \frac{1}{6} \frac{3}{4} = \frac{1}{8} \]

\[ P(AB) = \frac{1}{4} - \frac{1}{8} = \frac{1}{8} \]

\[ P(A|B) = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2} \]

(b) Let $A$ and $B$ be the same events as in part (a) and define $E = A - B$ and $F = B - A$. Are $E$ and $F$ independent? 

\[ A - B = AB^c, \quad B - A = BA^c \Rightarrow EF = \emptyset; \quad P(EF) = 0 \]

\[ P(AB^c) = \frac{1}{8} > 0, \quad P(BA^c) = P(B) - P(BA) = \frac{1}{4} - \frac{1}{8} = \frac{1}{8} > 0 \]

\[ E \text{ and } F \text{ are not independent} \]
(Problem 1 - cont’d)

Let $E$, $F$, and $G$ be events such that $P(E) = P(F) = P(G) = 1/3$, $P(EF) = P(EG) = P(FG) = 1/4$, and $P(EFG) = 1/5$. Find the probability that none of the events $E$, $F$, $G$ occur.

\[ p^* = P(\text{none of } E, F, G \text{ occur}) = P(\overline{EFG}) \\
= P((EUFUG)^c) = 1 - P(EUFUG) \]

By the inclusion-exclusion formula for 3 events

\[ P(EUFUG) = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG) \]

\[ = 3 \cdot \frac{1}{3} - 3 \cdot \frac{1}{4} + \frac{1}{5} \]

\[ = 1 - \frac{3}{4} + \frac{1}{5} = 1 - 0.75 + 0.2 \]

\[ = 0.45 \]

\[ p^* = 1 - P(EUFUG) = 1 - 0.45 = 0.55 \]
2. (a) How many different 8-place license plates are possible when 5 of the entries are letters (A-Z) and 3 are digits (0-9)? Assume that repetition of letters and numbers is allowed and that there is no restriction on where the letters or numbers can be placed. 

\[ \binom{8}{5} 26^5 \cdot 10^3 = 6,653,570,560,000 \]

(b) What is the number of all different license plates in part (a) if no repetition of numbers or letters is allowed?

\[ \binom{8}{5} 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 10 \cdot 9 \cdot 8 \]
\[ = \binom{8}{5} \frac{26!}{21! \cdot 10!} \]
\[ = \frac{3,182,679,952,000}{56} = 57,207,881,500 \]

(c) Assume we choose one license plate at random from the set of possible plates in part (b). What is the probability that the 3 digits will be consecutive and in increasing order (as in HG347JRK)?

\[ P = \frac{\binom{6}{3} \cdot \frac{26!}{21!}}{\binom{10}{3} \cdot \frac{26!}{21!}} = \frac{6 \cdot \frac{10!}{3! \cdot 21!}}{8 \cdot \frac{15 \cdot 25 \cdot 26}{3 \cdot 2 \cdot 1 \cdot 21!}} = \frac{6 \cdot 5!}{8 \cdot 7 \cdot 6} = \frac{1}{56} = 0.018 \]
Extra worksheet, if needed:
3. (a) A box contains 10,000 coins, one of which is double headed (has heads on both sides) and the rest are fair coins. A coin is chosen at random from the box and tossed 10 times. What is the (conditional) probability that it is the double headed coin if all 10 tosses have resulted in heads?

Let \( H_{10} \equiv \text{all 10 tosses are heads} \)

\[
P(DH | H_{10}) = \frac{P(H_{10} | DH)P(DH)}{P(H_{10} | DH)P(DH) + P(H_{10} | F)P(F)} = \frac{\frac{1}{10000}}{\frac{1}{10000} + \left(\frac{1}{2}\right)^{10} \frac{9999}{10000}} = \frac{1}{1 + 2 \times 9999}
\]

\[
\approx 0.093
\]

(b) In the same setup as in part (a), the following test is devised to decide if a double headed coin was chosen: The coin is tossed \( k \) times and if all tosses are heads, the test declares the coin double headed; otherwise it declares the coin fair. Find the minimum value of \( k \) such that the test will be correct with probability at least 0.999 whenever it declares the coin double-headed.

\[
P_k = \frac{\frac{1}{10000}}{\frac{1}{10000} + \left(\frac{1}{2}\right)^{k} \frac{9999}{10000}} = \frac{1}{1 + 2^{-k} \times 9999} \geq 0.999
\]

\[
1 + 2^{-k} \times 9999 \leq \frac{1}{0.999}
\]

\[
2^{-k} \leq \frac{1}{9999} \left(\frac{1}{0.999} - 1\right) \triangleq A
\]

\[
k \geq -\log_2 A = 23.25 \Rightarrow k_{\text{min}} = 24
\]
4. Suppose that on all airplanes operated by a certain airline, on any given flight each engine fails with probability $1/4$, independently of all other engines. Fortunately, an airplane can still fly to its destination if at least half of its engines work (but otherwise it crashes). Which kind of airplane has greater chance of reaching its destination, a 2-engine plane or a 4-engine plane? Is the answer different if the probability of engine failure is 0.4?

\[
P_k = \text{prob. that } k \text{ engine plane reaches destination}
\]

\[
P_2 = P(0 \text{ or } 1 \text{ engine fails}) = \left(\frac{3}{4}\right)^2 + \binom{2}{1} \left(\frac{1}{4}\right) \left(\frac{3}{4}\right)
\]

\[
= \frac{9}{16} + \frac{6}{16} = \frac{15}{16} = 0.9375
\]

\[
P_4 = P(0, 1, \text{or } 2 \text{ engines fail}) = \left(\frac{3}{4}\right)^4 + \binom{4}{1} \left(\frac{1}{4}\right) \left(\frac{3}{4}\right)^3 + \binom{4}{2} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^2
\]

\[
\approx 0.95
\]

Thus $P_4 > P_2$, it's better to fly the 4-engine plane.

For $P(\text{eng. failure}) = 0.4$:

\[
P_2 = (0.6)^2 + \binom{2}{1} (0.4)(0.6) = 0.84
\]

\[
P_4 = (0.6)^4 + \binom{4}{1} (0.4)(0.6)^3 + \binom{4}{2} (0.4)^2(0.6)^2
\]

\[
= 0.82
\]

\[
P_4 < P_2
\]

In this case, it is better to fly on the 2-engine plane.
5. A jar contains $N$ chips, numbered 1, 2, \ldots, $N$. Alice draws chips from the jar, one by one, at random, each time replacing the chip drawn before drawing the next. This continues until the first time it happens that she draws a chip that she has already drawn before. Let $X$ denote the total number of draws up to this point. For example, the sequence of outcomes could be (6, 1, 2, 3, 2), in which case, $X = 5$.

(a) Find the probability mass function of $X$. [5]

The maximum value of $X$ is $N+1$, thus $X = \{2, \ldots, N+1\}$.

For $k \in X$, let $B_{k-1} = \text{no rep. in the first } k-1 \text{ draws}$.

$$P(X = k) = P(B_{k-1}) P(\text{rep. on } k\text{th draw} \mid B_{k-1})$$

$$= \frac{N(N-1) \ldots (N-k+2)}{N^{k-1}} \cdot \frac{k-1}{N}$$

$$= \left[ \frac{N!}{(N-k+1)!} \cdot \frac{k-1}{N^k} , \quad k = 2, \ldots, N+1 \right]$$

(b) For $N = 3$, find the expected value and variance of $X$. Also, determine and sketch the distribution function of $X$. [5]

For $N = 3$, $X = \{2, 3, 4\}$

$$P(X = 2) = \frac{3 \cdot \frac{1}{3}}{3} = \frac{1}{3}$$

$$P(X = 3) = \frac{3 \cdot 2 \cdot \frac{2}{3}}{3^2} = \frac{4}{9}$$

$$P(X = 4) = \frac{3 \cdot 2 \cdot \frac{1}{3}}{3} = \frac{2}{9}$$

$$E(X) = 2 \cdot \frac{1}{3} + 3 \cdot \frac{4}{9} + 4 \cdot \frac{2}{9} = \frac{2.88}{3}$$

$$V_X(X) = E(X^2) - [E(X)]^2 = 2 \cdot \frac{1}{3} + 3 \cdot \frac{4}{9} + 4 \cdot \frac{2}{9} - (2.88)^2 \approx 0.544$$
Extra worksheet, if needed:

\[ F(x) = \begin{cases} 
0 & \text{if } x < 2 \\
\frac{1}{3} & \text{if } 2 \leq x < 3 \\
\frac{1}{3} + \frac{4}{9} = \frac{7}{9} & \text{if } 3 \leq x < 4 \\
1 & \text{if } x \geq 4
\end{cases} \]