Queen’s University  
Department of Mathematics and Statistics  

**STAT/MTHE 351**  
Midterm Examination   October 25, 2016

- Total points = 40. Duration = 2 hours.  
- Closed book, closed notes.  
- Simple calculators are permitted.  
- Write the answers in the space provided, continue on the backs of pages if needed.  
- **SHOW YOUR WORK CLEARLY.** Correct answers without clear work showing how you got there will not receive full marks.  
- Marks for each part of a question are shown in brackets at the right margin.

**Marks:** Please do not write in the space below.

Problem 1 [8]  
Problem 2 [8]  
Problem 3 [8]  
Problem 4 [8]  
Problem 5 [8]  

Total: [40]  

**Solutions**
1. (a) Let $A$ and $B$ be events, defined on some sample space, with $P(A) = 1/3$, $P(B) = 1/4$, and $P(AB) = 1/6$. Find the conditional probability $P(A^c|B)$.

\[
P(A^c|B) = 1 - P(A|B) = 1 - \frac{P(AB)}{P(B)}
\]

\[
= 1 - \frac{1/6}{1/4} = 1 - \frac{2}{3}
\]

\[
= \frac{1}{3}
\]

(b) For the same events as in part (a), find the probability that exactly one of the events $A$, $B$ occurs.

$C = \text{Exactly one of } A, B \text{ occurs}$

$= (A \cup B) - AB$

\[
P(C) = P((A \cup B) - AB) = P(A \cup B) - P(AB) \text{ (since } A \cap B \subset A \cup B)
\]

\[
= P(A) + P(B) - P(AB) - P(AB)
\]

\[
= \frac{1}{3} + \frac{1}{4} - \frac{1}{3} = \frac{1}{4}
\]
(Problem 1 - cont’d)

e) Suppose events $E$ and $F$ are independent and $G$ is an event that is disjoint from both $E$ and $F$. We know that $P(E) = 1/6$, $P(F) = 1/2$, and $P(G) = 1/3$. Find $P(E \cup F \cup G^c)$.

By the inclusion-exclusion formula

$$P(E \cup F \cup G^c) = P(E) + P(F) + P(G^c) - P(E \cap F) - P(E \cap G^c) - P(F \cap G^c) + P(E \cap F \cap G^c)$$

Where

$$P(G^c) = 1 - P(G) = \frac{2}{3}$$

$$P(E \cap F) = P(E)P(F) = \frac{1}{12} \text{ by independence}$$

$$P(E \cap G^c) = P(E) = \frac{1}{6} \text{ since } E \subset G^c$$

$$P(F \cap G^c) = P(F) = \frac{1}{2} \text{ since } F \subset G^c$$

$$P(E \cap F \cap G^c) = P(E) = \frac{1}{12}$$

Thus

$$P(E \cup F \cup G^c) = \frac{1}{6} + \frac{1}{2} + \frac{2}{3} - \frac{1}{12} - \frac{1}{6} - \frac{1}{2} + \frac{1}{12}$$

$$=\frac{2}{3}$$

Note: A simpler solution is obtained by noticing that since $E \subset G^c$ and $F \subset G^c$, we have $E \cup F \subset G^c \implies P(E \cup F \cup G^c) = P(G^c) = \frac{2}{3}$
2. Suppose an urn contains 10 white, 10 black, and 10 red balls. We pick six balls at random, without replacement. What is the probability there is at least one ball of each color among these six? [8]

(Hint: The complementary event is $\overline{W} \cup \overline{B} \cup \overline{R}$, where $\overline{W}$ is the event that no white ball is picked, $\overline{B}$ is the event that no black ball is picked, and $\overline{R}$ is the event that no red ball is picked.)

Let $W = \text{at least one white ball is picked}$

$B = \text{11 black}$

$R = \text{11 red}$

Then $E = \text{at least one of each color} = W \cup B \cup R = (W \cup B \cup R)^c$ so

$P(E) = 1 - P(W \cup B \cup R) = 1 - P(\overline{W}) - P(\overline{B}) - P(\overline{R})$ \[ + P(\overline{W}B) + P(\overline{W}R) + P(\overline{B}R) - P(\overline{W}B\overline{R}) \]

$P(\overline{W}) = P(\overline{B}) = P(\overline{R}) = \binom{20}{6} / \binom{30}{6}$

$P(\overline{W}B) = P(\overline{W}R) = P(\overline{B}R) = \binom{10}{6} / \binom{30}{6}$

$P(\overline{W}B\overline{R}) = P(\emptyset) = 0$

Thus

$\left[ P(E) = 1 - 3 \frac{\binom{20}{6}}{\binom{30}{6}} + 3 \frac{\binom{10}{6}}{\binom{30}{6}} \approx 0.805 \right]$
3. There are 2 fair and 3 double-headed coins in a box. A coin is randomly selected from
the box and is then repeatedly flipped.

(a) Let \( A \) and \( B \) the events that the first and second flips, respectively, result in heads.
Are \( A \) and \( B \) independent?

\[
P(A) = P(A | F)P(F) + P(A | D)P(D) \quad \text{(Law of total probability)}
\]
\[
= \frac{1}{2} \cdot \frac{2}{5} + 1 \cdot \frac{3}{5} = \frac{4}{5} = P(B)
\]
\[
P(AB) = \left(\frac{1}{2}\right)^2 \cdot \frac{2}{5} + 1 \cdot \frac{3}{5} = \frac{1}{10} + \frac{6}{10} = \frac{7}{10} \neq \left(\frac{4}{5}\right)^2
\]
Thus, \( A \) and \( B \) are not independent.

(b) If the first 5 flips all result in heads, what is the conditional probability that the coin
comes up heads on the 6th flip?

\[
P(E | G) = \frac{P(EG)}{P(G)}
\]
\[
P(EG) = P(EG | F)P(F) + P(EG | D)P(D)
\]
\[
= \left(\frac{1}{2}\right)^6 \cdot \frac{2}{5} + 1 \cdot \frac{3}{5}
\]
\[
P(G) = \left(\frac{1}{2}\right)^5 \cdot \frac{2}{5} + 1 \cdot \frac{3}{5}
\]
\[
P(E | G) = \frac{\left(\frac{1}{2}\right)^6 \cdot \frac{2}{5} + \frac{3}{5}}{\left(\frac{1}{2}\right)^5 \cdot \frac{2}{5} + \frac{3}{5}} = \frac{97}{98} \approx 0.99
\]
4. Two students must answer a true-false question. Each of them, independently of the other, gives the correct answer with probability $p$. If they are allowed to cooperate, which of the following is a better strategy for them? (a) Flip a fair coin to decide which student should answer the question; or (b) have them both consider the question and then either give the common answer if they agree or, if they disagree, flip a fair coin to determine which answer to give.

Let $A_i =$ student $i$ gives correct answer, $i = 1, 2$
$C =$ strategy gives correct answer

(a) $E_i =$ coin flip picks student $i$, $i = 1, 2$.

$$P(C) = \sum_{i=1}^{2} P(C \mid E_i)P(E_i) \quad \text{(law of total prob.)}$$
$$= \sum_{i=1}^{2} P(A_i)P(E_i) = 2 \cdot p \cdot \frac{1}{2} = \boxed{p}$$

(b) $C = A_1A_2 \cup A_1A_2^cE_1 \cup A_1^cA_2E_2$ (disjoint union)

Since $A_1, A_2, E_1, E_2$ are independent,

$$P(C) = P(A_1A_2) + P(A_1A_2^cE_1) + P(A_1^cA_2E_2)$$
$$= P(A_1)P(A_2) + P(A_1)P(A_2^c)P(E_1) + P(A_1^c)P(A_2)P(E_2)$$
$$= p^2 + 2p(1-p)\frac{1}{2} = p^2 + p(1-p) = \boxed{p}$$

Thus both strategies give the same probability for a correct answer.
5. Let $X$ be the number of vowels in the first three positions of a random permutation of the letters in the word "QUESTION". For example, "QUESTION" itself has two vowels in the first three positions, while "EUIQTOSN" has three vowels in the first three positions. Determine the probability mass function of $X$, i.e., the probabilities $P(X = k)$ for all possible values of $k$.

There are 4 vowels and 4 consonants in the word QUESTION. Thus, the possible values of $X$ are $X = \{0, 1, 2, 3\}$. Choose $S$ as the set of 3 letters in the first 3 positions. Then $|S| = \binom{8}{3}$ and each set of 3 letters is equally likely.

Thus

$$P(X = i) = \binom{4}{i} \binom{4}{3-i} \frac{\binom{8}{3}}{\binom{8}{3}}$$

for $i = 0, 1, 2, 3$.

$P(X = 0) = \frac{\binom{4}{0} \binom{4}{3}}{\binom{8}{3}} = \frac{4}{56} = \frac{1}{14} \approx 0.071$

$P(X = 1) = \frac{\binom{4}{1} \binom{4}{2}}{\binom{8}{3}} = \frac{24}{56} = \frac{6}{14} \approx 0.43$

$P(X = 2) = \frac{\binom{4}{2} \binom{4}{1}}{\binom{8}{3}} = \frac{24}{56} = \frac{6}{14} \approx 0.43$

$P(X = 3) = \frac{\binom{4}{3} \binom{4}{0}}{\binom{8}{3}} = \frac{1}{14} \approx 0.071$