Discrete Random Variables

We are often interested in the probabilistic description of how two or more random variables behave together. We will consider the case of two random variables.

**Definition** Suppose $X$ and $Y$ are two discrete r.v.’s defined on the same sample space, having possible values in the sets $\mathcal{X}$ and $\mathcal{Y}$, respectively. The joint probability mass function (pmf) of $X$ and $Y$ is defined by

$$p(x, y) = P(X = x, Y = y)$$

**Remark:** $P(X = x, Y = y)$ is a simplified notation for the probability

$$P(X = x \text{ and } Y = y) = P(\{X = x\} \cap \{Y = y\})$$

Given the joint pmf, we can recover the individual pmf’s of $X$ and $Y$:

$$p_X(x) = P(X = x) = P(X = x \text{ and } Y \in \mathcal{Y}) = \sum_{y \in \mathcal{Y}} P(X = x, Y = y) = \sum_{y \in \mathcal{Y}} p(x, y)$$

Thus

$$p_X(x) = \sum_{y \in \mathcal{Y}} p(x, y)$$

Similarly,

$$p_Y(y) = \sum_{x \in \mathcal{X}} p(x, y)$$

The pmf’s $p_X(x)$ and $p_Y(y)$ are called the marginal pmf’s of $X$ and $Y$. 

**Properties of the joint pmf**

1. $p(x, y) \geq 0$ for all $x$ and $y$.
2. $p(x, y) = 0$ if $x \notin \mathcal{X}$ or $y \notin \mathcal{Y}$.
3. $\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) = 1$

**Proof:** Follows the same lines as the proof of $\sum_{x \in \mathcal{X}} P(X = x) = 1$

4) For any $A \subset \mathbb{R}^2$,

$$P((X, Y) \in A) = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}; (x, y) \in A} p(x, y)$$

**Proof:** Follows the same lines as the proof of $P(X \in B) = \sum_{x \in \mathcal{X} : x \in B} P(X = x)$
**Example.** Five cards are drawn from a standard deck of 52 cards without replacement. Let $X$ be the number of spades drawn and $Y$ the number of clubs drawn. Find the joint pmf of $X$ and $Y$.

**Solution:** We have $X = Y = \{0, 1, 2, 3, 4, 5\}$. If $x + y > 5$, then $P(X = x, Y = y) = 0$. Otherwise

$$P(X = x, Y = y) = \frac{\binom{13}{x} \binom{13}{y} \binom{26}{5-x-y}}{\binom{52}{5}}$$

Thus

$$p(x, y) = \begin{cases} \frac{\binom{13}{x} \binom{13}{y} \binom{26}{5-x-y}}{\binom{52}{5}} & \text{if } 0 \leq x, y \leq 5, \ x + y \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

We obtain

<table>
<thead>
<tr>
<th>$y \backslash x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>$p_Y(y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1</td>
<td>0.05</td>
<td>0.05</td>
<td><strong>0.2</strong></td>
</tr>
<tr>
<td>1</td>
<td>0.3</td>
<td>0.2</td>
<td>0.1</td>
<td><strong>0.6</strong></td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>0.1</td>
<td>0</td>
<td><strong>0.2</strong></td>
</tr>
</tbody>
</table>

$p_X(x) = 0.5 \quad 0.35 \quad 0.15$

(This is why $p_X(x)$ and $p_Y(y)$ are called the marginal pmf’s.)

$$P(X + Y \leq 2) = \sum_{x+y \leq 2} p(x, y)$$

$$= p(0, 0) + p(0, 1) + p(1, 0) + p(1, 1) + p(0, 2) + p(2, 0)$$

$$= 0.1 + 0.3 + 0.05 + 0.2 + 0.1 + 0.05$$

$$= 0.8$$

**Example.** The joint pmf $p(x, y)$ of $X$ and $Y$ are given in the table below. Find the marginal pmf’s and $P(X + Y \leq 2)$.

<table>
<thead>
<tr>
<th>$y \backslash x$</th>
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<tbody>
<tr>
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<tr>
<td>2</td>
<td>0.1</td>
<td>0.07</td>
<td>0.03</td>
<td></td>
</tr>
</tbody>
</table>

$p_Y(0) = 0.2, \quad p_Y(1) = 0.6, \quad p_Y(2) = 0.2$

**Solution:**

$$p_X(0) = 0.1 + 0.3 + 0.1 = 0.5$$

$$p_X(1) = 0.05 + 0.2 + 0.1 = 0.35$$

$$p_X(2) = 0.05 + 0.1 + 0 = 0.15$$

Similarly,

$$p_Y(0) = 0.2, \quad p_Y(1) = 0.6, \quad p_Y(2) = 0.2$$

**Remark:** The joint pmf $p(x, y)$ *uniquely* determines the marginal pmf’s through the relationship

$$p_X(x) = \sum_{y \in \mathcal{Y}} p(x, y), \quad p_Y(y) = \sum_{x \in \mathcal{X}} p(x, y)$$

However, the reverse is *not true*. For given $p_X(x)$ and $p_Y(y)$ there are (infinitely) many joint pmf’s whose marginals are $p_X(x)$ and $p_Y(y)$.

For example, compare this table and the previous example:

<table>
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</table>

$p_X(x) = 0.5 \quad 0.35 \quad 0.15$
**Expected value**

Note that from the joint pmf we can compute the marginal pmf’s and so the expected values of $X$ and $Y$:

$$E(X) = \sum_{x \in X} xp_X(x) \text{ and } E(Y) = \sum_{y \in Y} yp_Y(y)$$

If $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a real function of two variables, $h(X, Y)$ is a discrete random variable.

**Theorem 1**

If $X$ and $Y$ are discrete r.v.’s with joint pmf $p(x, y)$ and $h(x, y)$ is a real function of two variables, then

$$E[h(X, Y)] = \sum_{x \in X} \sum_{y \in Y} h(x, y)p(x, y)$$

**Proof:** Follows the lines of the proof of $E[g(X)] = \sum_{x \in X} g(x)P(X = x)$.

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**Jointly Continuous Random Variables**

**Definition** The random variables $X$ and $Y$ are said to be *jointly continuous* if there exists a nonnegative function $f(x, y)$ such that for any "reasonable" planar set $C \subset \mathbb{R}^2$,

$$P((X, Y) \in C) = \int \int_C f(x, y) \, dx \, dy$$

The function $f(x, y)$ is called the *joint probability density function* (joint pdf) of $X$ and $Y$.

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**Corollary 2**

If $X$ and $Y$ are discrete r.v.’s, then

$$E(X + Y) = E(X) + E(Y)$$

**Proof:** With $h(x, y) = x + y$ we have

$$E(X + Y) = \sum_{x \in X} \sum_{y \in Y} (x + y)p(x, y)$$

$$= \sum_{x \in X} \sum_{y \in Y} xp(x, y) + \sum_{x \in X} \sum_{y \in Y} yp(x, y)$$

$$= \sum_{x \in X} x \sum_{y \in Y} p(x, y) + \sum_{y \in Y} y \sum_{x \in X} p(x, y)$$

$$= \sum_{x \in X} xp_X(x) + \sum_{y \in Y} yp_Y(y)$$

$$= E(X) + E(Y) \quad \square$$

---

**Properties of jointly continuous r.v.’s**

1. $f(x, y) \geq 0$ (by definition).
2. 

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1$$

**Proof:**

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = \int_{\mathbb{R}^2} f(x, y) \, dx \, dy$$

$$= P((X, Y) \in \mathbb{R}^2)$$

$$= 1 \quad \square$$
3. For any $a \leq b$ and $c \leq d$,

$$ P(a \leq X \leq b, c \leq Y \leq d) = \int_c^d \int_a^b f(x, y) \, dx \, dy $$

Proof: Let $C = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. Then

$$ P(a \leq X \leq b, c \leq Y \leq d) = P((X, Y) \in C) = \int_C f(x, y) \, dx \, dy = \int_c^d \int_a^b f(x, y) \, dx \, dy $$

Note: Letting $a = b = u$ and $c = d = v$, this shows that

$$ P(X = u, Y = v) = 0 $$

for all $u$ and $v$.

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Example: Let the joint pdf of $X$ and $Y$ be given by

$$ f(x, y) = \begin{cases} 
  kxy & 0 \leq x, y \leq 1 \\
  0 & \text{otherwise}
\end{cases} $$

(a) Determine $k$.

Solution:

$$ 1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = \int_0^1 \int_0^1 kxy \, dx \, dy = k \left( \int_0^1 x \, dx \right) \left( \int_0^1 y \, dy \right) = k \left( \frac{1}{2} \right)^2 = k \frac{1}{4} $$

Thus $k = 4$.

---

Similarly to the discrete case, the marginal pdf’s can be obtained from the joint pdf:

$$ f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy \quad \text{and} \quad f_Y(y) = \int_{-\infty}^{\infty} f(x, y) \, dx $$

Proof sketch:

$$ P(a \leq X \leq b) = P(a \leq X \leq b, -\infty < Y < \infty) = \int_{-\infty}^{\infty} \int_a^b f(x, y) \, dx \, dy = \int_a^b \left( \int_{-\infty}^{\infty} f(x, y) \, dy \right) \, dx $$

Thus the function $g(x) = \int_{-\infty}^{\infty} f(x, y) \, dy$ satisfies

$$ P(a \leq X \leq b) = \int_a^b g(x) \, dx $$

for all $a < b$, and so it must equal to the pdf $f_X$ of $X$.

(b) Find the marginal pdf’s.

Solution: For $0 \leq x \leq 1$,

$$ f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy = \int_0^1 4xy \, dy = \frac{2}{x} $$

Note that $f(x, y) = 0$ if $x \notin [0, 1]$, so $f_X(x) = 0$ if $x \notin [0, 1]$. Thus

$$ f_X(x) = \begin{cases} 
  2x & 0 \leq x \leq 1 \\
  0 & \text{otherwise}
\end{cases} $$

By symmetry

$$ f_Y(y) = \begin{cases} 
  2y & 0 \leq y \leq 1 \\
  0 & \text{otherwise}
\end{cases} $$
(c) Find \( P(X + Y \leq 1) \).

**Solution:** Note that \( P(X + Y \leq 1) = P((X, Y) \in C) \), where
\[
C = \{(x, y) : x + y \leq 1\}
\]
Thus
\[
P(X + Y \leq 1) = \iint_C f(x, y) \, dx \, dy = \iint_{x+y \leq 1} f(x, y) \, dx \, dy
\]
\[
= \int_0^1 \int_0^{1-y} 4xy \, dx \, dy
\]
\[
= \int_0^1 2y(1-y)^2 \, dy
\]
\[
= 2 \int_0^1 (y - 2y^2 + y^3) \, dy = 2 \left[ \frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4} \right]_0^1
\]
\[
= 2 \left( \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{1}{6}
\]

(b) Find the marginal pdf’s \( f_X(x) \) and \( f_Y(y) \).

**Solution:** For \( x \in [0, 1] \),
\[
f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy = \int_x^1 10xy^2 \, dy = \frac{10}{3}x(1 - x^3)
\]
Thus
\[
f_X(x) = \begin{cases} \frac{10}{3}x(1 - x^3) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}
\]
Similarly, for \( y \in [0, 1] \),
\[
f_Y(y) = \int_{-\infty}^{\infty} f(x, y) \, dx = \int_0^y 10xy^2 \, dx = 5y^4
\]
hence
\[
f_Y(y) = \begin{cases} 5y^4 & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}
\]

**Example:** Suppose the joint pdf of \( X \) and \( Y \) is given by
\[
f(x, y) = \begin{cases} 10xy^2 & 0 \leq x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}
\]
(a) Find \( P(Y \leq 2X) \).

**Solution:** Note that \( f(x, y) = 0 \) outside the region
\[
A = \{(x, y) : 0 \leq x \leq y \leq 1\}.
\]
Letting \( C = \{(x, y) : y \leq 2x \} \), we have
\[
P(Y \leq 2X) = \iint_C f(x, y) \, dx \, dy = \iint_{C \cap A} 10xy^2 \, dx \, dy
\]
\[
= \int_0^1 \int_0^{y/2} 10xy^2 \, dx \, dy
\]
\[
= 5 \int_0^1 y^2 \left( y^2 - \frac{y^2}{4} \right) \, dy
\]
\[
= \frac{15}{4} \int_0^1 y^4 \, dy = \frac{3}{4}
\]

**Geometric Probability**

Suppose a point is drawn randomly from a bounded planar region \( B \subset \mathbb{R}^2 \). The mathematical model is a pair of random variables \( (X, Y) \) with joint pdf given by
\[
f(x, y) = \begin{cases} c & \text{if } (x, y) \in B \\ 0 & \text{otherwise} \end{cases}
\]
where \( c > 0 \) is a constant.

The constant is determined by
\[
1 = \iint_{\mathbb{R}^2} f(x, y) \, dx \, dy = \iint_B c \, dx \, dy \quad \text{(since } f(x, y) = 0 \text{ outside } B)
\]
\[
= c \cdot \text{area}(B)
\]
so that \( c = \frac{1}{\text{area}(B)} \).
Let’s compute \( P((X, Y) \in A) \) for some \( A \subset \mathbb{R}^2 \):
\[
P((X, Y) \in A) = \int_A f(x, y) \, dx \, dy = \int_{A \cap B} \frac{1}{\text{area}(B)} \, dx \, dy = \frac{\text{area}(A \cap B)}{\text{area}(B)}
\]
Thus if \((X, Y)\) is a point randomly drawn from \(B\), then for all “reasonable” \(A \subset \mathbb{R}^2\),
\[
P((X, Y) \in A) = \frac{\text{area}(A \cap B)}{\text{area}(B)}
\]

### Expected value

Just as in the discrete case, using the joint pdf we can compute the marginal pdf’s and thus the expected values of \(X\) and \(Y\):
\[
E(X) = \int_{-\infty}^{\infty} x f_X(x) \, dx \quad \text{and} \quad E(Y) = \int_{-\infty}^{\infty} y f_Y(y) \, dy
\]

The following theorem shows how to calculate the expected value of the r.v. \(h(X, Y)\).

**Theorem 3**

If \(X\) and \(Y\) are continuous r.v.’s with joint pdf \(f(x, y)\) and \(h(x, y)\) is a real function of two variables, then
\[
E[h(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) f(x, y) \, dx \, dy
\]

**Example**: Let \((X, Y)\) be a point randomly drawn from the unit square \(B = \{(x, y) : 0 \leq x, y \leq 1\}\). Calculate \(P(|X - Y| \geq 1/2)\).

**Solution**: Note that \(\text{area}(B) = 1\). The probability can be calculated by integrating the pdf
\[
f(x, y) = \begin{cases} 
1 & \text{if } 0 \leq x, y \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]
over the region \(A = \{(x, y) : |x - y| \geq 1/2\}\).
However, since \(A \cap B\) is a union of two congruent isosceles right triangles of side length \(1/2\), we have
\[
P(|X - Y| \geq 1/2) = \frac{\text{area}(A \cap B)}{\text{area}(B)} = \frac{2 \cdot \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2}{1} = \frac{1}{4}
\]

**Corollary 4**

If \(X\) and \(Y\) are jointly continuous r.v.’s, then
\[
E(X + Y) = E(X) + E(Y)
\]

**Proof** Similar to the discrete case, but the sums are replaced with integrals. Letting \(h(x, y) = x + y\) we have
\[
E(X + Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x + y) f(x, y) \, dx \, dy
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y) \, dx \, dy + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x, y) \, dx \, dy
\]
\[
= \int_{-\infty}^{\infty} x \left( \int_{-\infty}^{\infty} f(x, y) \, dy \right) \, dx + \int_{-\infty}^{\infty} y \left( \int_{-\infty}^{\infty} f(x, y) \, dx \right) \, dy
\]
\[
= \int_{-\infty}^{\infty} x f_X(x) \, dx + \int_{-\infty}^{\infty} y f_Y(y) \, dy
\]
\[
= E(X) + E(Y)
\]
Remark: The main theorem about the expected value of \( h(X, Y) \) can be used for both discrete and jointly continuous r.v.'s to show the following “linearity of expectation” property:

\[
E[\alpha g_1(X, Y) + \beta g_2(X, Y)] = \alpha E[g_1(X, Y)] + \beta E[g_2(X, Y)]
\]

for any two functions \( g_1, g_2 : \mathbb{R}^2 \to \mathbb{R} \).

Example: Let \( X \) and \( Y \) have joint pdf

\[
f(x, y) = \begin{cases} 
10xy^2 & 0 \leq x \leq y \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

(a) Find \( E(X^2 + Y^2) \).

Solution: Using the linearity of expectation,

\[
E(X^2 + Y^2) = E(X^2) + E(Y^2)
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 f(x, y) \, dx \, dy + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^2 f(x, y) \, dx \, dy
\]

\[
= \int_{0}^{1} \int_{0}^{y} x^2 10xy^2 \, dx \, dy + \int_{0}^{1} \int_{0}^{y} y^2 10xy^2 \, dx \, dy
\]

\[
= \frac{5}{14} + \frac{5}{7} = \frac{15}{14}
\]

(b) Find \( E(X^2Y^2) \).

Solution:

\[
E(X^2Y^2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 y^2 f(x, y) \, dx \, dy
\]

\[
= \int_{0}^{1} \int_{0}^{y} x^2 y^2 10xy^2 \, dx \, dy
\]

\[
= 10 \int_{0}^{1} \int_{0}^{y} x^3 y^4 \, dx \, dy
\]

\[
= \frac{5}{18}
\]

Note that \( E(X^2)E(Y^2) = \frac{5}{14} \cdot \frac{5}{7} = \frac{25}{98} \), so that

\[
E(X^2Y^2) \neq E(X^2)E(Y^2)
\]

Independence of Random Variables

Definition Two random variables \( X \) and \( Y \) defined on the same probability space are independent if for all “reasonable” \( A, B \subset \mathbb{R} \), the events \( \{ X \in A \} \) and \( \{ Y \in B \} \) are independent, i.e.,

\[
P(X \in A, Y \in B) = P(X \in A)P(Y \in B)
\]

Let \( F(s, t) \) denote the joint distribution function of \( X \) and \( Y \):

\[
F(s, t) = P(X \leq s, Y \leq t)
\]

It can be shown using the axioms of probability that \( X \) and \( Y \) are independent if and only if

\[
F(s, t) = F_X(s)F_Y(t)
\]

i.e., the joint distribution function is the product of the marginal distribution functions.
The following theorem gives an easy to check characterization of independence for discrete random variables.

**Theorem 5**

Let $X$ and $Y$ be discrete r.v.’s with joint pmf $p(x, y)$. Then $X$ and $Y$ are independent if and only if

$$p(x, y) = p_X(x)p_Y(y)$$

i.e., the joint pmf is the product of the marginal pmf’s.

**Proof** If $X$ and $Y$ are independent, then $\{X = x\}$ and $\{Y = y\}$ are independent events, so $p(x, y) = p_X(x)p_Y(y)$ follows.

The proof that $p(x, y) = p_X(x)p_Y(y)$ for all $x, y$ implies that $X$ and $Y$ are independent is left as an exercise.

**Example:** If $X$ and $Y$ have pmf given below, are they independent?

<table>
<thead>
<tr>
<th>$y$</th>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>0.07</td>
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<td>0.07</td>
<td>0.03</td>
<td></td>
</tr>
</tbody>
</table>

**Solution:** They are independent. Calculate the marginal pmf’s and check that $p(x, y) = p_X(x)p_Y(y)$ for all $x$ and $y$:

<table>
<thead>
<tr>
<th>$y$</th>
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<td></td>
</tr>
</tbody>
</table>

$\sum p_X(x) = 0.5, \sum p_Y(y) = 0.2 = p(x, y)$

**Example:** The joint pmf of $X$ and $Y$ is given by

<table>
<thead>
<tr>
<th>$y$</th>
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<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1</td>
<td>0.05</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td>1</td>
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<td>0.2</td>
<td>0.1</td>
<td></td>
</tr>
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<td>0.1</td>
<td>0.1</td>
<td>0</td>
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</table>

Are $X$ and $Y$ independent?

**Solution:** The marginal pmf’s are

$p_X(0) = 0.5, p_X(1) = 0.35, p_X(2) = 0.15$

and

$p_Y(0) = 0.2, p_Y(1) = 0.6, p_Y(2) = 0.2$

Clearly, $X$ and $Y$ are not independent since e.g.,

$p_X(2)p_Y(2) = 0.15 \cdot 0.2 = 0.03 \neq 0 = p(2, 2)$

**Independence of continuous r.v.’s**

The following characterizes the independence of jointly continuous random variables in term of their pdf’s:

**Theorem 6**

Let $X$ and $Y$ be jointly continuous r.v.’s with joint pdf $f(x, y)$. Then $X$ and $Y$ are independent if and only if

$$f(x, y) = f_X(x)f_Y(y)$$

i.e., the joint pdf is the product of the marginal pdf’s.
Example: Let \( \Omega \) be the planar region defined by

\[
\Omega = \{(x, y) : 0 \leq x + y \leq 1, \ 0 \leq x \leq 1, \ 0 \leq y \leq 1\}
\]

and the joint pdf of \( X \) and \( Y \) be given by

\[
f(x, y) = \begin{cases} 
  6x & (x, y) \in \Omega \\
  0 & \text{otherwise}
\end{cases}
\]

Are \( X \) and \( Y \) independent?

Solution: Let’s calculate \( f_X(x) \) and \( f_Y(y) \):

\[
f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy = \begin{cases} 
  \int_0^x 6x \, dy = 6x(1 - x) & 0 \leq x \leq 1 \\
  0 & \text{otherwise}
\end{cases}
\]

and

\[
f_Y(y) = \int_{-\infty}^{\infty} f(x, y) \, dx = \begin{cases} 
  \int_0^1 6x \, dx = 3(1 - y)^2 & 0 \leq y \leq 1 \\
  0 & \text{otherwise}
\end{cases}
\]

Clearly, \( f(x, y) \neq f_X(x)f_Y(y) \). For example,

\[
f_X(x)f_Y(y) > 0 \quad \text{for all } (x, y) \in (0, 1)^2 - \Omega
\]

but

\[
f(x, y) = 0 \quad \text{if } (x, y) \in (0, 1)^2 - \Omega
\]

Thus \( X \) and \( Y \) are not independent.

Example: A point \((X, Y)\) is selected at random from the rectangle

\[
R = \{(x, y) : 0 \leq x \leq a, \ 0 \leq y \leq b\}
\]

Are \( X \) are \( Y \) independent?

Solution: Recall that the joint pdf is given by

\[
f(x, y) = \begin{cases} 
  \frac{1}{\text{area}(R)} = \frac{1}{ab} & \text{if } (x, y) \in R \\
  0 & \text{otherwise}
\end{cases}
\]

Thus

\[
f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy = \begin{cases} 
  \int_0^b \frac{1}{ab} \, dy = \frac{1}{a} & 0 \leq x \leq a \\
  0 & \text{otherwise}
\end{cases}
\]

Similarly

\[
f_Y(y) = \int_{-\infty}^{\infty} f(x, y) \, dx = \begin{cases} 
  \int_0^a \frac{1}{ab} \, dx = \frac{1}{b} & 0 \leq y \leq b \\
  0 & \text{otherwise}
\end{cases}
\]

Clearly, \( f(x, y) = f_X(x)f_Y(y) \) for all \( x \) and \( y \), so \( X \) and \( Y \) are independent.
Example: Let $X$ and $Y$ be independent exponential r.v.'s with pdf’s

$$f_X(x) = \begin{cases} \alpha e^{-\alpha x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad f_Y(y) = \begin{cases} \beta e^{-\beta y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(X < Y)$.

Solution: Since $X$ and $Y$ are independent,

$$f(x, y) = f_X(x)f_Y(y) = \begin{cases} \alpha \beta e^{-\alpha x-\beta y} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

We will calculate

$$P(X < Y) = \iiint_{(x, y): x < y} f(x, y) \, dx \, dy$$

Some consequences of independence

Theorem 7

Let $X$ and $Y$ be discrete or jointly continuous independent random variables and $h(x)$ and $g(y)$ real functions. Then $h(X)$ and $g(Y)$ are independent r.v.'s.

For example, if $X$ and $Y$ are independent, then $\sin X$ and $\cos Y$ are independent, $\cos X$ and $\sin Y$ are independent, $X^2$ and $e^{-Y}$ are independent, etc.

Theorem 8

Let $X$ and $Y$ be discrete or jointly continuous independent random variables and $h(x)$ and $g(y)$ real functions. Then

$$E[h(X)g(Y)] = E[h(X)]E[g(Y)]$$

Proof Assume $X$ and $Y$ are jointly continuous; the proof for the discrete case is similar:

$$E[h(X)g(Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x)g(y)f(x, y) \, dx \, dy$$

(by independence)

$$= \int_{-\infty}^{\infty} g(y) \left( \int_{-\infty}^{\infty} h(x)f_X(x) \, dx \right) dy$$

$$= E[h(X)] \int_{-\infty}^{\infty} g(y)f_Y(y) \, dy$$

$$= E[h(X)]E[g(Y)] \quad \square$$
Corollary 9

If \( X \) and \( Y \) are independent random variables, then

\[
E(XY) = E(X)E(Y)
\]

Remark: The converse of the corollary is not true, i.e., the fact that \( E(XY) = E(X)E(Y) \) does not imply that \( X \) and \( Y \) are independent.

We have

\[
E(X) = E(Y) = 0
\]

and

\[
E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x,y) \, dxdy
= \int_{R_1 \cup R_2 \cup R_3 \cup R_4} xy \, dxdy
= \sum_{i=1}^{4} \int_{R_i} xy \, dxdy
= \int_{1/2}^{1/2} \int_{0}^{1/2} xy \, dxdy + \int_{0}^{1/2} \int_{-1/2}^{1/2} xy \, dxdy
+ \int_{-1/2}^{0} \int_{-1/2}^{0} xy \, dxdy + \int_{-1/2}^{0} \int_{1/2}^{1/2} xy \, dxdy
= \left( \frac{1}{8} \right) \left( \frac{3}{8} \right) + \left( \frac{1}{8} \right) \left( -\frac{3}{8} \right) + \left( \frac{1}{8} \right) \left( -\frac{3}{8} \right) + \left( \frac{3}{8} \right) \left( -\frac{1}{8} \right) = 0
\]

Thus \( E(X)E(Y) = E(XY) \), but \( X \) and \( Y \) are not independent.

Example: Let \( R_1, R_2, R_3, \) and \( R_4 \) be the square regions defined by

\[
R_1 = \{ 0 \leq x \leq \frac{1}{2}, \frac{1}{2} \leq y \leq 1 \}, \quad R_2 = \{ -1 \leq x \leq -\frac{1}{2}, 0 \leq y \leq \frac{1}{2} \}
\]

\[
R_3 = \{ -\frac{1}{2} \leq x \leq 0, -1 \leq y \leq -\frac{1}{2} \}, \quad R_4 = \{ \frac{1}{2} \leq x \leq 1, -\frac{1}{2} \leq y \leq 0 \}
\]

and assume \((X,Y)\) is randomly drawn from \( A = R_1 \cup R_2 \cup R_3 \cup R_4 \). Then the joint pdf is

\[
f(x,y) = \begin{cases} 
\frac{1}{\text{area}(A)} = 1 & (x,y) \in A \\
0 & \text{otherwise}
\end{cases}
\]

It is easy to see that the marginal pdf’s are

\[
f_X(x) = \begin{cases} 
\frac{1}{2} & -1 \leq x \leq 1 \\
0 & \text{otherwise}
\end{cases}, \quad f_Y(y) = \begin{cases} 
\frac{1}{2} & -1 \leq y \leq 1 \\
0 & \text{otherwise},
\end{cases}
\]

Thus both \( X \) and \( Y \) are uniform r.v.’s on the interval \([-1,1]\).

Clearly, \( f(x,y) \neq f_X(x)f_Y(y) \), so \( X \) and \( Y \) are not independent.

Example: Let \( X \) and \( Y \) be independent uniform r.v.’s on the interval \((0,1)\). Find the pdf of \( Z = XY \) and use it to calculate \( E(Z) \).

Solution: The joint pdf \( f(x,y) = f_X(x)f_Y(y) \) is given by

\[
f(x,y) = \begin{cases} 
1 & 0 < x, y < 1 \\
0 & \text{otherwise}
\end{cases}
\]

Since \( Z = XY \) and \( X,Y \in (0,1) \), we have \( Z = XY \in (0,1) \), and so

\[
F_Z(t) = P(Z \leq t) = \begin{cases} 
0 & t \leq 0 \\
P(XY \leq t) & 0 < t < 1 \\
1 & t \geq 1
\end{cases}
\]
For $0 < t < 1$ we have from the geometry of the problem
\[
P(XY \leq t) = P\left( Y \leq \frac{t}{X} \right) = t \cdot 1 + \int_{t}^{1} \int_{0}^{t/x} dydx
\]
\[
= t + \int_{t}^{1} \frac{t}{x} dx = t - t \ln t
\]
Thus the pdf of $Z = XY$ is given by
\[
f_{Z}(t) = F_{Z}'(t) = \begin{cases}
\frac{d}{dt}(t - t \ln t) & 0 < t < 1 \\
0 & \text{otherwise}
\end{cases}
\]
\[
= \begin{cases}
-t \ln t & 0 < t < 1 \\
0 & \text{otherwise}
\end{cases}
\]
We can calculate $E(Z)$ as
\[
E(Z) = \int_{-\infty}^{\infty} tf_{Z}(t) dt = \int_{0}^{1} (-t \ln t) dt
\]
Using integration by parts with $u = -\ln t$ and $dv = t$, we obtain
\[
E(Z) = \int_{0}^{1} (-t \ln t) dt = \left[ -\frac{t^2}{2} \ln t \right]_{0}^{1} + \int_{0}^{1} \frac{t^2}{2} \cdot \frac{1}{t} dt
\]
\[
= 0 - 0 + \int_{0}^{1} \frac{t}{2} dt
\]
\[
= \frac{1}{4}
\]
Compare the above with the the following simple calculation which uses the fact that $E(XY) = E(X)E(Y)$ since $X$ and $Y$ are independent:
\[
E(XY) = E(X)E(Y) = \left( \int_{0}^{1} x dx \right) \left( \int_{0}^{1} y dy \right)
\]
\[
= \left( \frac{1}{2} \right)^2 = \frac{1}{4}
\]

Conditional Distributions

Recall the definition of conditional probability of an event $A$ given another event $B$ (such that $P(B) > 0$):
\[
P(A|B) = \frac{P(AB)}{P(B)}
\]
We want to extend this notion to a pair of random variables $X$ and $Y$. Specifically, we are interested in how the knowledge of the value of one of them "affects" the probability distribution of the other.

Discrete distributions

Let $X$ and $Y$ be discrete random variables with joint pmf $p(x, y)$. If we don’t know anything about the the value of $Y$, the probabilities concerning $X$ are calculated from the marginal pmf
\[
p_{X}(x) = \sum_{y \in Y} p(x, y)
\]
Now assume that we know that $Y = y$. Then we have extra knowledge about the probabilities concerning $X$ in the form of the conditional probabilities
\[
P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p(x, y)}{p_{Y}(y)}
\]
**Definition** Let $X$ and $Y$ be discrete r.v.’s with joint pmf $p(x, y)$. The conditional pmf of $X$ given $Y = y$ is defined by

$$p_{X|Y}(x|y) = \frac{p(x, y)}{p_Y(y)}$$

whenever $p_Y(y) > 0$.

**Note:** For fixed $y \in \mathcal{Y}$, the function $p_{X|Y}(x|y)$ is a pmf in $x$. Indeed, $p_{X|Y}(x|y) \geq 0$ and

$$\sum_{x \in \mathcal{X}} p_{X|Y}(x|y) = \sum_{x \in \mathcal{X}} \frac{p(x, y)}{p_Y(y)} = \frac{1}{p_Y(y)} \sum_{x \in \mathcal{X}} p(x, y) = \frac{1}{p_Y(y)} p_Y(y) = 1$$

**Example:** Let $X$ be the number of spades and $Y$ the number of clubs in a randomly drawn poker hand. We have seen that

$$p(x, y) = \begin{cases} \binom{13}{x} \binom{13}{y} \binom{26}{5-x-y} \\ 0 \end{cases} \frac{1}{\binom{52}{5}}$$

if $0 \leq x, y \leq 5, \ x + y \leq 5$

otherwise

Let’s calculate the conditional pmf of $X$ given $Y$. We have

$$p_Y(y) = \frac{\binom{13}{y} \binom{39}{52-y}}{\binom{52}{5}}, \quad 0 \leq y \leq 5$$

Thus

$$p_{X|Y}(x|y) = \frac{p(x, y)}{p_Y(y)} = \frac{\binom{13}{x} \binom{13}{y} \binom{26}{5-x-y} / \binom{52}{5}}{\binom{y}{y} \binom{39}{52-y} / \binom{52}{5}} = \frac{\binom{13}{x} \binom{26}{5-x-y} / \binom{39}{52-y}}{\binom{5}{5-x-y}} \quad 0 \leq x \leq 5 - y$$

**Remarks:**

1. The conditional pmf $p_{Y|X}(y|x)$ is similarly defined.
2. The conditional pmf can be used to calculate the conditional probability of events in the form $\{X \in C\}$ given $\{Y = y\}$:

$$P(X \in C|Y = y) = \sum_{x \in C} p_{X|Y}(x|y)$$

**Proof** This is essentially the law of total probability. The formal proof is left as an exercise.

3. If $X$ and $Y$ are independent, then $p(x, y) = p_X(x)p_Y(y)$, so

$$p_{X|Y}(x|y) = \frac{p(x, y)}{p_Y(y)} = \frac{p_X(x)p_Y(y)}{p_Y(y)} = p_X(x)$$

Thus in this case $p_{X|Y}(x|y) = p_X(x)$. The converse is also true, and in fact $X$ and $Y$ are independent if and only if $p_{X|Y}(x|y) = p_X(x)$ for all $x$ and $y$ such that $p_Y(y) > 0$.
Continuous distributions

It is not immediately clear how to condition on the value of a continuous random variable \( Y \) since \( P(Y = y) = 0 \) for all \( y \).

**Definition** Let \( X \) and \( Y \) be jointly continuous r.v.’s with joint pdf \( f(x, y) \). The **conditional pdf** of \( X \) given \( Y = y \) is defined by

\[
f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}
\]

whenever \( f_Y(y) > 0 \). Similarly, the conditional pdf of \( Y \) given \( X = x \) is

\[
f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)}
\]

whenever \( f_X(x) > 0 \).

**Remarks:**

1. Just as in the discrete case, it is easy to show that \( f_{X|Y}(x|y) \) is a valid pdf for fixed \( y \):

\[
\int_{-\infty}^{\infty} f_{X|Y}(x|y) \, dx = \int_{-\infty}^{\infty} \frac{f(x, y)}{f_Y(y)} \, dx = \frac{1}{f_Y(y)} \int_{-\infty}^{\infty} f(x, y) \, dx = \frac{1}{f_Y(y)} f_Y(y) = 1
\]

2. Similarly to the discrete case, it can be shown that \( X \) and \( Y \) are independent if and only if

\[
f_{X|Y}(x|y) = f_X(x)
\]

for all \( x \) and \( y \) such that \( f_Y(y) > 0 \).

**Definition** For a reasonable set \( B \subset \mathbb{R} \), the conditional probability of the event \( \{ Y \in B \} \) given \( X = x \) is defined by

\[
P(Y \in B|X = x) = \int_B f_{Y|X}(y|x) \, dy
\]

**Note:** The meaning of \( P(Y \in B|X = x) \) is not as obvious as in the discrete case, since we are conditioning on the event \( \{ X = x \} \) of probability zero. We will see later how useful \( P(Y \in B|X = x) \) can be.

**Example:** Let the joint pdf of \( X \) and \( Y \) be

\[
f(x, y) = \begin{cases} 10xy^2 & 0 \leq x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}
\]

(a) Find \( f_{X|Y}(x|y) \) and \( f_{Y|X}(y|x) \).

**Solution:** We have seen that

\[
f_Y(y) = \begin{cases} 5y^4 & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}
\]

We have \( f_Y(y) > 0 \) if \( y \in (0, 1) \). Thus for all \( y \in (0, 1) \),

\[
f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \begin{cases} \frac{10xy^2}{5y^4} = \frac{2x}{y^2} & 0 \leq x < y \\ 0 & \text{otherwise} \end{cases}
\]
Previously we have also calculated $f_X(x)$:

$$f_X(x) = \begin{cases} 10x(1-x^3) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus for $x \in (0, 1)$ (so that $f_X(x) > 0$),

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \begin{cases} \frac{10xy^2}{10x(1-x^3)} = \frac{3y^2}{1-x^3} & x \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Find $P(X \leq \frac{1}{2} | Y = \frac{3}{4})$.

**Solution:** We have

$$f_{X|Y}(x|3/4) = \begin{cases} \frac{2x}{(\frac{3}{4})^2} & 0 \leq x \leq \frac{3}{4} \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$P(X \leq \frac{1}{2} | Y = \frac{3}{4}) = \int_{-\infty}^{1/2} f_{X|Y}(x|y) \, dx$$

$$= \frac{32}{9} \int_{0}^{1/2} x \, dx$$

$$= \frac{32}{9} \cdot \frac{1}{8} = \frac{4}{9}$$

The conditional probability $P(Y \in B | X = x)$ can be very useful in calculating probabilities via the following version of the law of total probability:

**Theorem 10 (Law of total probability)**

Let $X$ and $Y$ be discrete random variables. Then

$$P(Y \in B) = \sum_{x \in X} P(Y \in B | X = x)p_X(x)$$

If $X$ and $Y$ are jointly continuous random variables, then

$$P(Y \in B) = \int_{-\infty}^{\infty} P(Y \in B | X = x) f_X(x) \, dx$$

**Proof** We only do the continuous case:

$$\int_{-\infty}^{\infty} P(Y \in B | X = x) f_X(x) \, dx$$

$$= \int_{-\infty}^{\infty} \left( \int_{B} f_{Y|X}(y|x) \, dy \right) f_X(x) \, dx$$

$$= \int_{B} \left( \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x) \, dx \right) dy$$

$$= \int_{B} f_Y(y) \, dy$$

$$= P(Y \in B) \quad \square$$
**Example:** Suppose $X$ is uniformly distributed on $[0, 1]$. Given $X = x$, let $Y$ be a random point in the interval $[0, x]$. Calculate the probability $P(Y \geq 1/2)$.

**Solution:** We could calculate $f_{X,Y}(x, y)$ and obtain $P(Y \geq 1/2)$ from a double integral. Instead, we will use the law of total probability.

We have

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

and so

$$P(Y \geq 1/2|X = x) = \begin{cases} \frac{x - 1/2}{x} & 1/2 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus from the law of total probability we have

$$P(Y \geq 1/2) = \int_{-\infty}^{\infty} P(Y \geq 1/2|X = x) f_X(x) \, dx$$

$$= \int_{1/2}^{1} \frac{x - 1/2}{x} \, dx$$

$$= \int_{1/2}^{1} \left(1 - \frac{1}{2x}\right) \, dx$$

$$= \left[ x - \frac{1}{2} \ln x \right]_{1/2}^{1}$$

$$= \frac{1}{2} (1 - \ln 2)$$

**Conditional Expectation**

We can generalize the expected value to conditional distributions in the following way.

**Definition** Let $X$ and $Y$ be random variables defined on the same sample space. If $X$ and $Y$ are discrete, then the conditional expectation of $X$ given $Y = y$ is defined by

$$E(X|Y = y) = \sum_{x \in \mathcal{X}} x p_{X|Y}(x|y)$$

whenever $p_Y(y) > 0$. If $X$ and $Y$ are jointly continuous, the corresponding definition is

$$E(X|Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) \, dx$$

whenever $f_Y(y) > 0$.

**Note:** For fixed $y$, the conditional expectation of $X$ given $Y = y$ is simply an expectation calculated according to the conditional distribution of $X$ given $Y = y$.

Thus all the basic properties we derived for (unconditional) expectations still hold. For example,

$$E(h(X)|Y = y) = \sum_{x \in \mathcal{X}} h(x) p_{X|Y}(x|y)$$

for discrete r.v.’s, and

$$E(h(X)|Y = y) = \int_{-\infty}^{\infty} h(x) f_{X|Y}(x|y) \, dx$$

for continuous r.v.’s.
**Example.** Let the joint pdf of $X$ and $Y$ be

$$f(x, y) = \begin{cases} 
10xy^2 & 0 \leq x \leq y \leq 1 \\
0 & \text{otherwise}
\end{cases}$$

(a) Find $E(X|Y = y)$.

**Solution:** We previously derived

$$f_{X|Y}(x|y) = \begin{cases} 
\frac{2x}{y^2} & 0 \leq x < y \leq 1 \\
0 & \text{otherwise}
\end{cases}$$

Thus for all $y \in [0, 1]$ we need to calculate

$$E(X|Y = y) = \int_{-\infty}^{\infty} xf_{X|Y}(x|y) \, dx = \int_{0}^{y} x \frac{2x}{y^2} \, dx$$

(b) Find $E(Y|X = x)$.

**Solution:** From previous calculations,

$$f_{Y|X}(y|x) = \begin{cases} 
\frac{3y^2}{1 - x^3} & 0 \leq x < y \leq 1 \\
0 & \text{otherwise}
\end{cases}$$

Hence for $x \in [0, 1]$

$$E(Y|X = x) = \int_{-\infty}^{\infty} yf_{Y|X}(y|x) \, dy = \int_{x}^{1} \frac{3y^2}{1 - x^3}$$

Conditional expectation can greatly simplify the calculation of expected value.

**Theorem 11 (Law of total expectation)**

Let $X$ and $Y$ be random variables defined on the same sample space. If $X$ and $Y$ are discrete, then

$$E(Y) = \sum_{x \in X} E(Y|X = x)p_X(x)$$

If $X$ and $Y$ are jointly continuous, then

$$E(Y) = \int_{-\infty}^{\infty} E(Y|X = x)f_X(x) \, dx$$

**Note:** The proof is very similar to the proof of the law of total probability and is left as an exercise. The theorem also holds if the roles of $X$ and $Y$ are exchanged.
Example: Suppose $X$ is uniformly distributed on $[0, 1]$. Given $X = x$, let $Y$ be a random point in the interval $[0, x]$. Calculate $E(Y)$.

Solution: Since $Y$ is uniformly distributed on $[0, x]$ given $X = x$, we have

$$E(Y|X = x) = \frac{x}{2}$$

Thus by the law of total expectation,

$$E(Y) = \int_0^\infty E(Y|X = x) f_X(x) \, dx = \int_0^1 \frac{x}{2} \, dx = \frac{1}{4}$$

Exercise: Calculate $E(Y)$ by finding $f_Y(y)$ first. Which solution is simpler?

Transformations of Two Random Variables

Here we consider the two-dimensional generalization of the problem of finding the pdf of $h(X)$ for an invertible $h : \mathbb{R} \to \mathbb{R}$ and continuous r.v. $X$.

Suppose $X$ and $Y$ are jointly continuous and $h_1, h_2 : \mathbb{R}^2 \to \mathbb{R}$ are real functions. We want to determine the joint pdf of the pair of r.v.’s $U = h_1(X, Y)$, $V = h_2(X, Y)$

We will be able to do this if $h_1$ and $h_2$ satisfy certain regularity conditions.

Let $\mathcal{J}$ denote the Jacobian of $(g_1, g_2)$, i.e.,

$$\mathcal{J}(u, v) = \det \begin{bmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \end{bmatrix} = \frac{1}{\mathcal{J}(g_1(u, v), g_2(u, v))}$$

Theorem 12 (Joint pdf of transformed random variables)

Assume $X$ and $Y$ have joint pdf $f_{X,Y}(x, y)$. If $h_1$ and $h_2$ satisfy the above conditions, then the joint pdf of $U = h_1(X, Y)$ and $V = h_2(X, Y)$ is given by

$$f_{U,V}(u, v) = f_{X,Y}(g_1(u, v), g_2(u, v)) |\mathcal{J}(u, v)|$$

Remark: The proof of the theorem relies on the change of variable formula for double integrals.
**Example:** Let $X$ and $Y$ be independent standard normal r.v.'s. Let $R$ and $\Theta$ be the polar coordinates of the point $(X, Y)$, i.e.,

$$R = \sqrt{X^2 + Y^2}, \quad \Theta = \arctan(Y/X)$$

Find the joint pdf of $R$ and $\Theta$.

**Solution:** We have $h_1(x, y) = \sqrt{x^2 + y^2}$ and $h_2(x, y) = \arctan(y/x)$. We have to solve

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan(y/x)$$

for $x$ and $y$. The solution (of course) is the well-known expression

$$x = r \cos \theta = g_1(r, \theta), \quad y = r \sin \theta = g_2(r, \theta)$$

Note that $-\infty < x < \infty$ and $-\infty < y < \infty$, while $0 \leq r < \infty$ and $0 \leq \theta < 2\pi$.

Since $X$ and $Y$ are independent,

$$f_{X,Y}(x, y) = f_X(x)f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$$= \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

Therefore

$$f_{R,\Theta}(r, \theta) = f_{X,Y}(g_1(r, \theta), g_2(r, \theta)) |J(r, \theta)|$$

$$= f_{X,Y}(r \cos \theta, r \sin \theta) |J(r, \theta)|$$

$$= \frac{1}{2\pi} e^{-r^2/2} r$$

for all $r > 0$ and $0 \leq \theta < 2\pi$.

The partial derivatives of $g_1$ and $g_2$ are

$$\frac{\partial g_1}{\partial r} = \cos \theta, \quad \frac{\partial g_1}{\partial \theta} = -r \sin \theta$$

and

$$\frac{\partial g_2}{\partial r} = \sin \theta, \quad \frac{\partial g_2}{\partial \theta} = r \cos \theta$$

Thus the Jacobian of $(g_1, g_2)$ is

$$J(r, \theta) = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r$$

since $\cos^2 \theta + \sin^2 \theta = 1$.

In conclusion, we obtained that

$$f_{R,\Theta}(r, \theta) = \begin{cases} \frac{1}{2\pi} r e^{-r^2/2} & r > 0, 0 \leq \theta < 2\pi \\ 0 & \text{otherwise} \end{cases}$$

The marginal pdf's are

$$f_R(r) = \begin{cases} r e^{-r^2/2} & r > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \leq \theta < 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Since $f_{R,\Theta}(r, \theta) = f_R(r)f_\Theta(\theta)$, the random variables $R$ and $\Theta$ are independent. Also note that $\Theta$ is uniformly distributed on $[0, 2\pi)$.
Sum of two independent random variables

Assume $X$ and $Y$ are independent and let $U = X + Y$. We want to find the pdf of $U$ in terms of the pdf's of $X$ and $Y$.

One way to do this is to calculate the joint pdf $f_{U,V}(u,v)$ of the pair $U = X + Y$, $V = X - Y$, and then find $f_U(u)$ as the marginal of $f_{U,V}(u,v)$.

Instead, we follow a more direct approach:

$$F_U(u) = P(U \leq u) = \int_{x+y \leq u} f_{X,Y}(x,y) \, dxdy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{u-y} f_X(x)f_Y(y) \, dx\,dy$$

Thus

$$f_U(u) = F_U'(u) = \frac{d}{du} \int_{-\infty}^{\infty} \int_{-\infty}^{u-y} f_X(x)f_Y(y) \, dx\,dy$$

$$= \int_{-\infty}^{\infty} \frac{d}{du} \left( \int_{-\infty}^{u-y} f_X(x) \, dx \right) f_Y(y) \, dy$$

$$= \int_{-\infty}^{\infty} f_X(u-y) f_Y(y) \, dy$$

We obtained

$$f_U(u) = \int_{-\infty}^{\infty} f_X(u-y) f_Y(y) \, dy$$

By symmetry, the same proof also implies that

$$f_U(u) = \int_{-\infty}^{\infty} f_X(x) f_Y(u-x) \, dx$$

Remark: The integral

$$\int_{-\infty}^{\infty} f_X(u-y) f_Y(y) \, dy$$

is called the convolution of $f_X(x)$ and $f_Y(y)$ and is denoted by $f_X * f_Y$.

Example: Let $X$ and $Y$ be independent uniform r.v.'s on $[0,1]$. Find the pdf of $U = X + Y$.

Solution: We have $f_U = f_X * f_Y$, where

$$f_X(t) = f_Y(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$f_U(u) = \int_{-\infty}^{\infty} f_X(u-y) f_Y(y) \, dy = \int_{0}^{1} f_X(u-y) \, dy$$

If $u < 0$ or $u > 2$, then $f_U(u) = 0$. For $0 \leq u < 1$ we have

$$\int_{0}^{1} f_X(u-y) \, dy = \int_{0}^{u} dy = u$$

For $1 \leq u \leq 2$,

$$\int_{0}^{1} f_X(u-y) \, dy = \int_{u-1}^{1} dy = 2 - u$$

Putting these together, we obtain

$$f_U(u) = \begin{cases} u & 0 \leq u < 1 \\ 2 - u & 1 \leq u \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

This function is called the triangular pdf.