Introduction

What is probability? Simple answer: analysis and calculation of chance(s).

The outcomes of many observable (natural) phenomena cannot be predicted with certainty. A method of quantifying “likelihood” is needed.

Gambling is the most obvious example, and was the main motivation in the early development of probability theory.

Numerical calculation of risks, chances, etc. is desirable in real life and in many fields of science:
- Stock market
- Randomized surveys
- Biology, medicine, genetics, pharmacology
- Physics, chemistry, engineering, etc.

Most people have an intuitive understanding of what probability is. E.g.:

“If an unbiased coin is flipped, there is a 50% chance of heads coming up.”

But the subjective interpretation of probability can vary person to person.

To develop a theory of chances that withstands scientific scrutiny, we need an axiomatic approach.

The mathematical theory of probability is formally a branch of mathematics. However, it is unique in that intuition plays a larger than usual role in formalizing, solving, and interpreting problems.

This course is a (gentle) introduction to probability theory.

Nevertheless, this is a mathematics course with definitions, theorems, and proofs!
Quick overview of basic set theory

- A set $A$ is a collection of objects from the universal set $S$. No object is repeated; the order of objects in $A$ does not matter.
- The members of a set are called its elements. The notation $a \in A$ means that $a$ is an element of $A$. The notation $a \notin A$ means that “$a$ is not an element of $A”.$

Means of describing (defining) sets:

**Listing of elements** Useful if a set has only a few elements. E.g.,

$A = \{1, 2, 3, 4, 5, 6\}$, or $B = \{\text{cat, dog}\}$

**Defining rules** Usual form is

$A = \{x : x \text{ satisfies some property}\}$

For example,

$A = \{j : j \text{ is an integer between 1 and 100}\}$

**Venn diagrams** ...

Examples:

- Empty set (null set). Contains no elements. Denoted by $\emptyset$.
- The set of integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
- The set of real numbers $\mathbb{R}$.
- The set of real numbers between 0 and 1 (both inclusive)

$[0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$.
- The set of real numbers strictly between 0 and 1

$(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$.
- The set of all possible outcomes of two rolls of a die is compactly represented as $\{(i, j) : i, j = 1, 2, 3, 4, 5, 6\}$. Contains 36 elements.

Subsets $A$ is called a *subset* of $B$, if every element of $A$ is also in $B$.

The notation is $A \subset B$. In such a case, we also informally say that $A$ is contained in $B$.

![Venn diagram](image)

$A \subset B$.

Note: $B \supset A$ is equivalent to $A \subset B$.

Equality $A = B$ if and only if $A \subset B$ and $B \subset A$.

Complementation The *complement* of a set $A$ is the set of all elements of $S$ that are *not* in $A$. The notation is $A^c$. Thus,

$A^c = \{x \in S : x \notin A\}$.

We usually simply write $A^c = \{x : x \notin A\}$.

The shaded region is the complement of $A$. 

**Set union** The union of $A$ and $B$, denoted as $A \cup B$, is the set of elements that belong to at least one of the two sets. Thus,

$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ (or both)}\}.$$ 

$A \cup B$ is the shaded region.

*Note:* $A \cup B = B \cup A$.

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**Set difference** The difference of $A$ and $B$, $A - B$ (sometimes denoted as $A \setminus B$) is the set of elements of $A$ that are not in $B$. Thus

$$A - B = \{x : x \in A : x \notin B\} = AB^c.$$ 

Note that $A^c = S - A$.

$A - B$ is the shaded region.

*Note:* $A - B \neq B - A$ in general.

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**Intersection** The intersection of $A$ and $B$, denoted as $A \cap B$ or $AB$, is the set of elements that belong to both the sets.

$$AB = \{x : x \in A \text{ and } x \in B\}.$$ 

$AB$ is the shaded region.

*Note:* $AB = BA$.

$A$ and $B$, are (mutually) *disjoint* if they have no common elements, i.e., if $AB = \emptyset$. We’ll often say “mutually exclusive” instead of “disjoint.”

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**Cartesian products** The Cartesian product of $A$ and $B$ is the set $A \times B$ defined as follows:

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$ 

Note that in general $A \times B \neq B \times A$.

For example, if $A = \{1\}$ and $B = \{0, 1\}$, then

$$A \times B = \{(1, 0), (1, 1)\}$$

while

$$B \times A = \{(0, 1), (1, 1)\}$$

Sometimes we write $A^2$ for $A \times A$. For example

$$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$$
Properties of set operations

- \( S^c = \emptyset \)
- \( (A^c)^c = A \)
- \( A \cup \emptyset = A, \quad A \cap \emptyset = \emptyset \)
- \( A \cup S = S, \quad A \cap S = A \)
- \( A \cup A^c = S, \quad AA^c = \emptyset \)

Commutativity:

\[
A \cup B = B \cup A \\
AB = BA
\]

Associativity:

\[
(A \cup B) \cup C = A \cup (B \cup C) \\
(AB)C = A(BC)
\]

Distributivity:

\[
A \cup (BC) = (A \cup B)(A \cup C) \\
A(B \cup C) = AB \cup AC
\]

DeMorgan’s laws:

\[
(A \cup B)^c = A^c B^c \\
(AB)^c = A^c \cup B^c
\]