Assignment 2 — due Friday, Feb. 2

1. Let $X_1, \ldots, X_n$ be a set of independent and identically distributed continuous random variables.
   (a) Show that the random vector $(X_n, X_{(n)})^T$ does not have a joint pdf.
   (b) For $1 \leq k \leq n - 1$, compute $P(\max(X_1, \ldots, X_k) \leq \min(X_{k+1}, \ldots, X_n))$.

2. Let $X_1, X_2, X_3$ be independent, identically distributed continuous random variables. Find the probability that the second largest value (i.e., the median) is closer to the smallest value than to the largest value, when the common distribution of the $X_i$ is
   (a) the Uniform$(0,1)$ distribution (a symmetry argument should suffice here);
   (b) the Exponential$(\lambda)$ distribution.

3. Let $X_1, \ldots, X_n$ be a sequence of independent Uniform$(0,1)$ random variables, with $X_{(1)}, \ldots, X_{(n)}$ denoting their order statistics. Let $A_n$ denote the expected area of the triangle formed by the vertices $(X_{(n-2)}, 0)$, $(X_{(n-1)}, X_{(n-1)})$, and $(X_n, 0)$. Find $A_n$ (in terms of $n$) and show that $n A_n \to 1$ as $n \to \infty$.

4. Let $X_1, \ldots, X_n$ be mutually independent Uniform$(0,1)$ random variables. Find the probability that the interval $(\min(X_1, \ldots, X_n), \max(X_1, \ldots, X_n))$ contains the value $1/2$ and find the smallest $n$ such that this probability is at least 0.95.

5. Let $X_1, \ldots, X_n$ be a random sample of size $n$ from an exponential distribution with parameter $\lambda > 0$, and let $X_{(1)}, \ldots, X_{(n)}$ denote the order statistics. Find
   \[
   E\left[ e^{t_1 X_{(1)} + t_2 X_{(2)}} \right]
   \]
   for all values of $t_1, t_2$ for which the expectation exists.