Problem 1. (5 marks) There are 3 categories for the error that we are interested in: less than 2 millimeters (category 1), between 2 and 3 millimeters (category 2), and greater than 3 millimeters (category 3). Let $X_1, X_2, X_3$ denote, respectively, the number of errors (out of 10) that are in categories 1, 2 and 3. Then $(X_1, X_2, X_3)$ has a Multinomial distribution with parameters $n = 10$ and $p_1, p_2, p_3$, where

$$p_1 = P\text{(error is less than 2mm)} = \Phi\left(\frac{2}{2}\right) = 0.841$$

$$p_2 = P\text{(error is between 2mm and 3mm)} = \Phi\left(\frac{3}{2}\right) - \Phi\left(\frac{2}{2}\right) = 0.933 - 0.841 = 0.092$$

$$p_3 = P\text{(error is greater than 3mm)} = 1 - \Phi\left(\frac{3}{2}\right) = 0.067.$$ 

The event of interest, that at least 9 measurements have error less than 2 millimeters and no more than 1 measurement has error greater than 3 millimeters, corresponds to the outcomes \{ $X_1 = 9, X_2 = 0, X_3 = 1$ \} or \{ $X_1 = 9, X_2 = 1, X_3 = 0$ \} or \{ $X_1 = 10, X_2 = 0, X_3 = 0$ \}. Let $A$ denote the event of interest. Then

$$P(A) = P(X_1 = 9, X_2 = 0, X_3 = 1) + P(X_1 = 9, X_2 = 1, X_3 = 0) + P(X_1 = 10, X_2 = 0, X_3 = 0)$$

$$= \frac{10!}{9!0!1!} p_1^9 p_2^0 p_3^1 + \frac{10!}{9!1!0!} p_1^9 p_2^1 p_3^0 + \frac{10!}{10!0!0!} p_1^{10} p_2^0 p_3^0$$

$$= (10)(.841)^9(.067) + (10)(.841)^9(.092) + (.841)^{10} = 0.512.$$ 

Problem 2. (7 marks)

(a) (2 marks) Let $X_i$ be the number of $i$'s that are rolled, for $i = 1, \ldots, 6$. Then $(X_1, \ldots, X_6)$ has a Multinomial distribution with parameters 10 and $p_i = 1/6$ for $i = 1, \ldots, 6$, and the desired probability is

$$P(X_1 = 1, X_2 = 2, X_3 = 3, X_4 = 4) = P(X_1 = 1, X_2 = 2, X_3 = 3, X_4 = 4, X_5 = 0, X_6 = 0)$$

$$= \frac{10!}{1!2!3!4!0!0!} \left(\frac{1}{6}\right)^{10} \approx 0.0002084.$$
(b) (2 marks) Let \(X_{12}\) be the number of 1’s or 2’s, \(X_3\) the number of 3’s, and \(X_{456}\) the number of 4’s, 5’s or 6’s. Then the vector \((X_{12}, X_3, X_{456})\) has a Multinomial distribution with parameters 10 and \(p_{12} = 1/3, p_3 = 1/6, p_{456} = 1/2\). The desired probability is
\[
P(X_{12} = 3, X_3 = 4) = P(X_{12} = 3, X_3 = 4, X_{456} = 3) = \frac{10!}{3!4!3!} \left( \frac{1}{3} \right)^3 \left( \frac{1}{6} \right)^4 \left( \frac{1}{2} \right)^3 \approx 0.015.
\]

(c) (3 marks) Let \(X_{123}\) be the number of 1’s, 2’s, or 3’s, \(X_{46}\) the number of 4’s or 6’s, and \(X_5\) the number of 5’s. Then \((X_{123}, X_{46}, X_5)\) has a Multinomial distribution with parameters 10 and \(p_{123} = 1/2, p_{46} = 1/3, p_5 = 1/6\). The desired probability is
\[
P(X_5 = 3 \mid X_{123} = 4) = \frac{P(X_{46} = 3, X_5 = 3 \mid X_{123} = 4)}{P(X_{123} = 4)} = \frac{6!}{3!3!} \left( \frac{1/3}{1/3 + 1/6} \right)^3 \left( \frac{1/6}{1/3 + 1/6} \right)^3 = \frac{6!}{3!3!} \left( \frac{2}{3} \right)^3 \left( \frac{1}{3} \right)^3 \approx 0.2195.
\]

Problem 3. (6 marks) We have
\[
P\left(X_1 = x_1, \ldots, X_k = x_k \mid \sum_{i=1}^k X_i = n\right) = \frac{P\left(X_1 = x_1, \ldots, X_k = x_k, \sum_{i=1}^k X_i = n\right)}{P\left(\sum_{i=1}^k X_i = n\right)}.
\]
First, we note that the above conditional probability is nonzero if and only if each of \(x_1, \ldots, x_n\) is a nonnegative integer and \(\sum_{i=1}^n x_i = n\) (since otherwise the numerator on the right hand side above is 0). For \(x_1, \ldots, x_k\) nonnegative integers satisfying \(\sum_{i=1}^k x_i = n\), we have
\[
\frac{P\left(X_1 = x_1, \ldots, X_k = x_k, \sum_{i=1}^k X_i = n\right)}{P\left(\sum_{i=1}^k X_i = n\right)} = \frac{P\left(X_1 = x_1, \ldots, X_k = x_k\right)}{P\left(\sum_{i=1}^k X_i = n\right)} = \frac{P\left(X_1 = x_1\right) \ldots P\left(X_k = x_k\right)}{P\left(\sum_{i=1}^k X_i = n\right)} = \frac{\left(\lambda_1^{x_1}/x_1!\right) e^{-\lambda_1} \ldots \left(\lambda_k^{x_k}/x_k!\right) e^{-\lambda_k}}{P\left(\sum_{i=1}^k X_i = n\right)}.
\]
Now if we multiply and divide the last expression above by \(\frac{n!}{(\lambda_1 + \ldots + \lambda_k)^n}\), we get that the conditional probability is equal to
\[
P\left(X_1 = x_1, \ldots, X_k = x_k \mid \sum_{i=1}^k X_i = n\right) = \frac{e^{-\left(\lambda_1 + \ldots + \lambda_k\right)} \left(\lambda_1 + \ldots + \lambda_k\right)^n}{n!P\left(\sum_{i=1}^k X_i = n\right)} \times \frac{n!}{x_1! \ldots x_k!} p_1^{x_1} \ldots p_k^{x_k},
\]
where \(p_i = e^{-\lambda_i}/\lambda_i!)\) for \(i = 1, \ldots, k\).
where \( p_j = \frac{\lambda_j}{\sum_{i=1}^{k} \lambda_i} \), for \( j = 1, \ldots, k \). Summing over all \((x_1, \ldots, x_k)\) satisfying \( \sum_{i=1}^{k} x_i = n \), we obtain

\[
1 = \sum_{(x_1, \ldots, x_k): \sum x_i = n} P\left( X_1 = x_1, \ldots, X_k = x_k \bigg| \sum_{i=1}^{k} X_i = n \right) \\
= \sum_{(x_1, \ldots, x_k): \sum x_i = n} \frac{e^{-(\lambda_1 + \cdots + \lambda_k)}(\lambda_1 + \cdots \lambda_k)^n}{n!P\left( \sum_{i=1}^{k} X_i = n \right)} \times \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k} \\
= \frac{e^{-(\lambda_1 + \cdots + \lambda_k)}(\lambda_1 + \cdots \lambda_k)^n}{n!P\left( \sum_{i=1}^{k} X_i = n \right)} \sum_{(x_1, \ldots, x_k): \sum x_i = n} \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k} \\
= \frac{e^{-(\lambda_1 + \cdots + \lambda_k)}(\lambda_1 + \cdots \lambda_k)^n}{n!P\left( \sum_{i=1}^{k} X_i = n \right)}
\]

(the sum is equal to 1 since the values \( n! \frac{x_1! \cdots x_k!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k} \) are probabilities from a Multinomial distribution with parameters \( n \) and \( p_1, \ldots, p_k \) and the sum is over the entire support of this distribution). Thus we conclude that

\[
P\left( X_1 = x_1, \ldots, X_k = x_k \bigg| \sum_{i=1}^{k} X_i = n \right) = \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}
\]

That is, the conditional distribution of \((X_1, \ldots, X_k)\) given \( \sum_{i=1}^{k} X_i = n \) is Multinomial with parameters \( n \) and \( p_1, \ldots, p_k \).

**Problem 4.** (5 marks) Label the 39 non-heart cards 1, \ldots, 39. Let

\[
X_i = \begin{cases} 
1 & \text{if the } i\text{th non-heart card is drawn before the first heart} \\
0 & \text{otherwise}.
\end{cases}
\]

Then \( X = 1 + X_1 + \cdots + X_{39} \) is the number of cards drawn until the first heart and

\[
E[X] = 1 + \sum_{i=1}^{39} P(\text{ith non-heart card is drawn before the first heart}).
\]

For concreteness, we can, for example, compute the probability that the ace of spades is drawn before the first heart (see Exercise 9, Section 3.2). The probability will be the same for every other non-heart card. Since every permutation of the 52 cards is equally likely we can count the number of permutations in which the ace of spades precedes every heart, and divide that number by 52!. First, note that any such permutation cannot have the ace of spades in a position larger than 39. For \( i = 1, \ldots, 39 \), let \( N_i \) be the number of permutations
where the ace of spades is in position $i$ and there are no hearts in positions $1, \ldots, i-1$. For positions $1, \ldots, i-1$ we first choose $i-1$ of the 38 remaining non-heart cards and then we can permute these in $(i-1)!$ ways. We can also permute the remaining $52-i$ cards into positions $i+1, \ldots, 52$ in $(52-i)!$ ways. Thus,

$$N_i = \binom{38}{i-1} (i-1)! (52-i)! = \frac{38! (52-i)!}{(39-i)!}.$$ 

Therefore, the probability that the ace of spades is drawn before any heart is

$$E[X_i] = \sum_{i=1}^{39} \frac{38! (52-i)!}{52! (39-i)!} = \frac{1}{52} + \frac{38}{52(51)} + \frac{(38)(37)}{(52)(51)(50)} + \frac{(38)(37)\ldots(1)}{(52)(51)\ldots(14)} = \frac{1}{52} \left( 1 + \frac{38}{51} \left( 1 + \frac{37}{50} \left( \ldots + \frac{1}{15} \left( 1 + \frac{2}{14} \left( 1 + \frac{1}{13} \left( \ldots + \frac{1}{1} \right) \right) \right) \right) \right).$$

$$E[X] = 1 + \frac{39}{14} = \frac{53}{14} \approx 3.786.$$

**Problem 5.** (5 marks) For $i = 1, \ldots, n-1$, let

$$X_i = \begin{cases} 1 & \text{if the } i\text{th adjacent pair of magnets repel each other} \\ 0 & \text{if the } i\text{th adjacent pair of magnets join.} \end{cases}$$

Then $X = 1 + X_1 + \ldots + X_{n-1}$ is the number of blocks. Since each pair of adjacent magnets repel each other with probability $1/2$, we have $E[X_i] = 1/2$. Therefore,

$$E[X] = 1 + E[X_1] + \ldots + E[X_{n-1}] = 1 + \frac{n-1}{2} = \frac{n+1}{2}.$$
We may compute the variance of $X$ as $\text{Var}(X) = E[X^2] - E[X]^2 = E[X^2] - (n+1)^2/4$. Now, $E[X^2]$ will involve terms of the form $E[X_iX_j]$, where $i \neq j$. If $i$ and $j$ are disjoint pairs then $E[X_iX_j] = E[X_i]E[X_j] = 1/4$ because $X_i$ and $X_j$ are independent in this case. If $j = i + 1$ then

$$E[X_iX_{i+1}] = P(3 \text{ consecutive magnets form 3 blocks}) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4},$$

as well. We have

$$E[X^2] = E[(1 + X_1 + \ldots + X_{n-1})^2] = 1 + 2\sum_{i=1}^{n-1} X_i + \sum_{i=1}^{n-1} X_i^2 + \sum_{i \neq j} X_iX_j = 1 + 2\frac{n-1}{2} + \frac{n-1}{2} + \frac{(n-1)(n-2)}{4} = \frac{n^2 + 3n}{4}.$$

and

$$\text{Var}(X) = E[X^2] - E[X]^2 = \frac{n^2 + 3n}{4} - \frac{(n+1)^2}{4} = \frac{n-1}{4}.$$