1. [Show that the Gamma\((r, \lambda)\) . . .]

Differentiating \(f(x)\) with respect to \(x\), we have

\[
f'(x) = \frac{\lambda^r}{\Gamma(r)} \left( (r-1)x^{r-2}e^{-\lambda x} - \lambda x^{r-1}e^{-\lambda x} \right) = \frac{\lambda^r}{\Gamma(r)} x^{r-2}e^{-\lambda x} (r - 1 - \lambda x).
\]

The term outside the parentheses in the final expression above is positive for all \(x > 0\). If \(r > 1\), then \(f'(x) > 0\) for \(0 < x < (r-1)/\lambda\), \(f'(x) < 0\) for \(x > (r-1)/\lambda\), and \(f'(x) = 0\) for \(x = (r-1)/\lambda\). Hence, \(f(x)\) is maximized at \(x = (r-1)/\lambda\).

Since \(\lambda > 0\), if \(r \leq 1\), the term inside the parentheses is negative for all \(x > 0\), so \(f'(x) < 0\) for all \(x > 0\) if \(r \leq 1\). Thus for \(r = 1\), \(f(x)\) is maximized at \(x = 0\). For \(r \in (0, 1)\), we have that \(\lim_{x \to 0^+} f(x) = \infty\), to \(x = 0\) can again be considered to maximize \(f(x)\).

2. [Let \(X\) and \(Y\) be . . .] The probability that the point \((X, Y)\) is at a distance of more than \(1.5\sigma\) from the origin is

\[
P(\sqrt{X^2 + Y^2} > 1.5\sigma) = P(X^2 + Y^2 > 2.25\sigma^2) = P \left( \frac{X^2 + Y^2}{\sigma^2} > 2.25 \right).
\]

Since \(X/\sigma\) and \(Y/\sigma\) are independent \(N(0, 1)\) random variables, we have (from results in class) that \(X^2/\sigma^2\) and \(Y^2/\sigma^2\) are independent \(\chi^2\) random variables with 1 degree of freedom. Therefore (also from results in class), \(X^2/\sigma^2 + Y^2/\sigma^2\) has a \(\chi^2\) distribution with 2 degrees of freedom, which is the same as an exponential distribution with parameter \(1/2\). Therefore,

\[
P(\sqrt{X^2 + Y^2} > 1.5\sigma) = P \left( \frac{X^2 + Y^2}{\sigma^2} > 2.25 \right) = e^{-2.25/2} = 0.3246.
\]

3. [Let \(X\) have Gamma\((m/2, 1/2)\) . . .]

We use the bivariate change of variable formula to get the joint density of \(U\) and \(V\). We have

\[
U = h_1(X, Y) = \frac{X}{m} \quad \text{and} \quad V = h_2(X, Y) = Y
\]

and the inverse transformation is

\[
X = w_1(U, V) = \frac{mUV}{n} \quad \text{and} \quad Y = w_2(U, V) = V.
\]
The Jacobian of the transformation is

\[ J = \det \begin{bmatrix} \frac{\partial w_1(u,v)}{\partial u} & \frac{\partial w_1(u,v)}{\partial v} \\ \frac{\partial w_2(u,v)}{\partial u} & \frac{\partial w_2(u,v)}{\partial v} \end{bmatrix} = \det \begin{bmatrix} \frac{mv}{n} & \frac{mu}{n} \\ 0 & 1 \end{bmatrix} = \frac{mv}{n}. \]

Since \( X \) and \( Y \) are independent, the joint density of \( X \) and \( Y \) is given by

\[ f(x, y) = \begin{cases} \frac{(1/2)^{m/2}}{\Gamma(m/2)} x^{(m/2)-1} e^{-x/2} \frac{(1/2)^{n/2}}{\Gamma(n/2)} y^{(n/2)-1} e^{-y/2} & \text{for } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases} \]

It is not hard to see that the support of \((U, V)\) is

\[ S_{U,V} = \{(u, v) \in \mathbb{R}^2 : u > 0, v > 0\}. \]

By the bivariate change of variable formula, the joint density of \( U \) and \( V \), say \( g(u,v) \), is computed as

\[
\begin{align*}
g(u,v) &= f(w_1(u,v), w_2(u,v)) |J| \\
&= f(mu/n,v) \frac{mv}{n} \\
&= \begin{cases} \\
\left(\frac{(1/2)^{m/2}}{\Gamma(m/2)} \frac{mu}{n}\right)^{(m/2)-1} e^{-mu/(2n)} \frac{(1/2)^{n/2}}{\Gamma(n/2)} v^{(n/2)-1} e^{-v/2} & \text{for } u > 0 \text{ and } v > 0 \\
0 & \text{otherwise} \\
\end{cases} \\
&= \begin{cases} \\
\frac{(1/2)^{(m+n)/2}}{\Gamma(m/2) \Gamma(n/2)} u^{(m/2)-1} v^{((m+n)/2)-1} e^{-(mu/(2n)+1/2)v} & \text{for } u > 0 \text{ and } v > 0 \\
0 & \text{otherwise} \\
\end{cases}
\end{align*}
\]

Now we integrate \( g(u,v) \) over \( v \) to get the marginal density function of \( U \), say \( g_U(u) \). For fixed \( u > 0 \), \( g(u,v) \) as a function of \( v \) is proportional to the pdf of a Gamma distribution with parameters \( m + n \) and \( \frac{2n}{2n+1} \). Therefore, computing the integral \( \int_0^\infty g(u,v) dv \) can be done by multiplying \( g(u,v) \) by the appropriate normalizing constant (which may depend on \( u \). For \( u > 0 \), we have

\[
\begin{align*}
g_U(u) &= \int_0^\infty g(u,v) dv \\
&= \left(\frac{1/2}{\Gamma(m/2) \Gamma(n/2)}\right)^{(m/2)-1} \int_0^\infty v^{((m+n)/2)-1} e^{-(mu/(2n)+1/2)v} dv \\
&= \left(\frac{1/2}{\Gamma(m/2) \Gamma(n/2)}\right)^{(m/2)-1} \frac{\Gamma((m+n)/2)}{(\frac{mu}{2n} + \frac{1}{2})^{(m+n)/2}} \times \int_0^\infty \frac{\left(\frac{mu}{2n} + \frac{1}{2}\right)^{(m+n)/2}}{\Gamma((m+n)/2)} v^{((m+n)/2)-1} e^{-(mu/(2n)+1/2)v} dv \\
&= \left(\frac{1/2}{\Gamma(m/2) \Gamma(n/2)}\right)^{(m/2)-1} \frac{\Gamma((m+n)/2)}{(\frac{mu}{2n} + \frac{1}{2})^{(m+n)/2}} \times \frac{m}{nB(m/2,n/2)} \cdot \frac{(mu/n)^{(m-2)/2}}{[1 + (mu/n)]^{(m+n)/2}} \\
&= \frac{m}{nB(m/2,n/2)} \cdot \frac{(mu/n)^{(m-2)/2}}{[1 + (mu/n)]^{(m+n)/2}},
\end{align*}
\]
where
\[ B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \quad \text{for } \alpha > 0, \beta > 0 \]
is the Beta function. Clearly, \( g_U(u) = 0 \) for \( u < 0 \) since \( g(u, v) = 0 \) for \( u < 0 \) and for any \( v \).

4. [Ghahramani, 7.5, #8]

(a) We have
\[ E[Y] = a + (b - a)E[X] = a + (b - a)\frac{\alpha}{\alpha + \beta} \]
and
\[ \text{Var}(Y) = (b - a)^2\text{Var}(X) = \frac{(b - a)^2\alpha\beta}{(\alpha + \beta + 1)(\alpha + \beta)^2}. \]

(b) The pdf of \( Y \) can be obtained from the pdf of \( X \) using the change of variable formula (Sec.6.2). We have
\[ y = h(x) = a + (b - a)x \text{ with inverse transformation } x = w(y) = (y - a)/(b - a), \]
and \( w'(y) = 1/(b - a) \). It’s easy to see that the support of \( Y \) is \([a, b]\). Therefore, letting \( f(x) \) denote the density function of \( X \) (the beta density with parameters \( \alpha \) and \( \beta \)), we have
\[
g(y) = f\left(\frac{(y - a)/(b - a)}{w'(y)}\right) \left| w'(y) \right|
= \frac{1}{(b - a)B(\alpha, \beta)} \left( \frac{y - a}{b - a} \right)^{\alpha - 1} \left( \frac{b - y}{b - a} \right)^{\beta - 1}
= \frac{1}{(b - a)^{\alpha + \beta - 1}B(\alpha, \beta)} \left( y - a \right)^{\alpha - 1} \left( b - y \right)^{\beta - 1},
\]
for \( a \leq y \leq b \) and \( g(y) = 0 \) otherwise, where \( B(\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \) is the beta function.

(c) With \( \alpha = 2, \beta = 3, a = 2 \) and \( b = 6 \), we wish to compute \( P(Y < 3) \). We have
\[
P(Y < 3) = P(2 + 4X < 3) = P(X < 0.25)
= \int_{0}^{0.25} \frac{4!}{1!2!} x(1 - x)^2 dx
= 12 \int_{0}^{0.25} (x - 2x^2 + x^3) dx
= 12 \left[ \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right]_{0}^{0.25}
= 12 \left[ \frac{0.0625}{2} - \frac{2 \times 0.015625}{3} + \frac{0.00390625}{4} \right]
= 12(0.03125 - 0.0104167 + 0.0009765625) \approx 0.262.
\]

3
5. [Let $X$ have a Gamma($r, \lambda$)…]

Define $U = XY$ and $V = X$. We use the bivariate change of variable formula to obtain the joint pdf of $U$ and $V$, then use this to obtain the marginal pdf of $U$. We have

$$U = h_1(X,Y) = XY \quad \text{and} \quad V = h_2(X,Y) = X$$

with inverse

$$X = w_1(U,V) = V \quad \text{and} \quad Y = w_2(U,V) = U/V.$$  

The Jacobian of the transformation is

$$J = \det \begin{bmatrix} \frac{\partial w_1(u,v)}{\partial u} & \frac{\partial w_1(u,v)}{\partial v} \\ \frac{\partial w_2(u,v)}{\partial u} & \frac{\partial w_2(u,v)}{\partial v} \end{bmatrix} = \det \begin{bmatrix} 0 & 1 \\ 1/v & -u/v^2 \end{bmatrix} = -1/v.$$  

The support of $(U,V)$ is

$$S_{U,V} = \{(u,v) \in \mathbb{R}^2 : 0 < u < v < \infty\}.$$  

The joint density of $X$ and $Y$ is

$$f(x,y) = \begin{cases} \frac{\lambda^{\alpha+\beta} e^{-\lambda x} \Gamma(\alpha+\beta)}{\Gamma(\alpha) \Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1} & \text{for } x > 0 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$  

By the bivariate change of variable formula, the joint density of $U$ and $V$, say $g(u,v)$, is computed as

$$g(u,v) = f(w_1(u,v), w_2(u,v))|J|$$

$$= f(v, u/v) \frac{1}{v}$$

$$= \begin{cases} \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha) \Gamma(\beta)} v^{\alpha+\beta-1} e^{-\lambda x} \Gamma(\alpha+\beta) y^{\alpha-1}(1-y)^{\beta-1} & \text{for } 0 < u < v < \infty \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha) \Gamma(\beta)} u^{\alpha-1} v^{\beta-1} e^{-\lambda v} & \text{for } 0 < u < v < \infty \\ 0 & \text{otherwise} \end{cases}$$

and for $u > 0$, the marginal density of $U$ is

$$g_U(u) = \int_{-\infty}^{\infty} g(u,v) dv = \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha) \Gamma(\beta)} u^{\alpha-1} \int_u^{\infty} (v-u)^{\beta-1} e^{-\lambda v} dv. $$  

Making the change of variable $t = v-u$, we have $v = t + u$, $dv = dt$, and

$$g_U(u) = \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha) \Gamma(\beta)} u^{\alpha-1} \int_0^{\infty} t^{\beta-1} e^{-\lambda(t+u)} dt$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} u^{\alpha-1} e^{-u} \int_0^{\infty} \frac{\lambda^{\beta}}{\Gamma(\beta)} t^{\beta-1} e^{-\lambda t} dt$$
\[ \frac{\lambda^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\lambda u}, \]

since the integral above is 1 as the integrand is the pdf the Gamma(\(\beta, \lambda\)) distribution. Clearly, 
\(g_U(u) = 0\) for \(u < 0\). Therefore, \(U = XY\) has a Gamma distribution with parameters \(\alpha\) and \(\lambda\), as
was to be shown.