Queen’s University  
Department of Mathematics and Statistics  

MTHE/STAT 353  
Midterm Examination  
February 26, 2016

• Total points = 30. Duration = 58 minutes.
• This is a closed book exam.
• One 8.5 by 11 inch sheet of notes, written on both sides, is permitted.
• A simple calculator is permitted.
• Write the answers in the space provided, continue on the backs of pages if needed.
• SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks.
• Marks per part question are shown in brackets at the right margin.
• The last page contains formulas you may find useful. Please check this page first.

Marks: Please do not write in the space below.

Problem 1 [10]

Problem 2 [10]

Problem 3 [10]

Total: [30] ___________
1. Let $X$ and $Y$ be jointly discrete random variables with joint pmf

$$p(x, y) = K \frac{(\lambda(1-p))^y \left(\frac{p}{1-p}\right)^x}{x!(y-x)!},$$

for $y = 0, 1, 2, \ldots$ and $x = 0, 1, \ldots, y$, and $p(x, y) = 0$ otherwise. Here $\lambda > 0$ and $p \in (0, 1)$ are both fixed parameters, and $K$ is a normalizing constant.

(a) Find $K$. [5]

Solution: We first sum over $x$ for fixed $y$, then sum over $y$. For fixed $y$ the terms in the sum over $x$ are binomial probabilities:

$$\sum_{y=0}^{\infty} \sum_{x=0}^{y} p(x, y) = K \sum_{y=0}^{\infty} \lambda^y \sum_{x=0}^{y} \frac{(1-p)^y \left(\frac{p}{1-p}\right)^x}{x!(y-x)!}$$

$$= K \sum_{y=0}^{\infty} \lambda^y \sum_{x=0}^{y} \frac{p^x (1-p)^{y-x}}{x!(y-x)!}$$

$$= K \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} \sum_{x=0}^{y} \left(\frac{y}{x}\right) p^x (1-p)^{y-x}$$

$$= K \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = Ke^\lambda,$$

where each sum over $x$ in the third equality is equal to one. Setting this equal to one gives $K = e^{-\lambda}$. 
(b) Find the marginal pmf of $X$. [5]

Solution: The possible values for $X$ are $x = 0, 1, 2, \ldots$. With $K = e^{-\lambda}$ as computed in part(a), summing $p(x, y)$ over $y$ for a fixed $x$ we compute the marginal pmf of $X$ at $x$ as

$$p_X(x) = \sum_{y=x}^{\infty} p(x, y)$$

$$= e^{-\lambda} \sum_{y=x}^{\infty} \frac{\lambda^y (1 - p)^{y-x}}{y!}$$

$$= e^{-\lambda} \sum_{y=x}^{\infty} \frac{(\lambda (1 - p))^{y-x}}{y!}$$

$$= e^{-\lambda} \sum_{y=0}^{\infty} \frac{(\lambda (1 - p))^y}{y!}$$

$$= e^{-\lambda} (\lambda (1 - p)) x! e^{\lambda (1 - p)}$$

$$= \frac{(\lambda p)^x}{x!} e^{-\lambda p},$$

which holds for $x = 0, 1, 2, \ldots$ and $p_X(x) = 0$ otherwise (i.e., the marginal distribution of $X$ is Poisson($\lambda p$)).
2. A system has 4 components. When power is applied to the system it takes some time for each component to initialize. Let $X_i$ be the time, in seconds, for component $i$ to initialize, for $i = 1, 2, 3, 4$. Suppose the $X_i$ are independent, continuous random variables, each with pdf

$$f_X(x) = \begin{cases} 
3x^2/b^3 & \text{for } 0 \leq x \leq b \\
0 & \text{otherwise,}
\end{cases}$$

where $b > 0$ is fixed. The system fails to initialize if and only if 2 or more components take longer than $b/2$ seconds to initialize. Find the probability that the system fails to initialize.

**Solution:** We wish to find $P(X_3 > b/2)$, where $X_3$ is the third order statistic of $X_1, X_2, X_3, X_4$ (i.e., $X_3$ is the third smallest, or the second largest, of the $X_i$’s). We first need the df of the $X_i$’s, which is

$$F_X(x) = \begin{cases} 
x^3/b^3 & \text{for } 0 \leq x \leq b \\
0 & \text{for } x < 0 \\
1 & \text{for } x > 1.
\end{cases}$$

Then the pdf of $X_3$ is

$$f_3(x_3) = 4! \frac{F_X(x_3)^2}{2!} f_X(x_3)(1 - F_X(x_3))$$

$$= 12 \left( \frac{x_3^3}{b^3} \right)^2 \frac{3x_3^2}{b^3} \left( 1 - \frac{x_3^3}{b^3} \right)$$

$$= 36 \frac{x_3^8}{b^9} - 36 \frac{x_3^{11}}{b^{12}}$$

for $0 \leq x_3 \leq b$, and $f_3(x_3) = 0$ otherwise. Then $P(X_3 > b/2)$ is computed as

$$P(X_3 > b/2) = \int_{b/2}^{b} \left( 36 \frac{x_3^8}{b^9} - 36 \frac{x_3^{11}}{b^{12}} \right) dx_3$$

$$= \left[ \frac{4x_3^9}{b^9} - \frac{3x_3^{12}}{b^{12}} \right]_{b/2}^{b}$$

$$= 1 - \left( \frac{4}{2^9} - \frac{3}{2^{12}} \right)$$

$$= 1 - \frac{29}{4096} = 1 - \frac{29}{4096} = 1 - 0.007 = 0.993.$$
3. Let $X_1, \ldots, X_{10}$ be independent Uniform$(0,1)$ random variables.

(a) Find the probability that all the $X_i$’s are greater than 0.5. [5]

Solution: This can be computed as

$$P(X_1 > .5, \ldots, X_{10} > .5) = P(X_1 > .5) \times \ldots \times P(X_{10} > .5)$$

by independence

$$= P(X_1 > .5)^{10}$$

as the $X_i$’s are identically distributed

$$= (.5)^{10} = \frac{1}{1024} = 0.000977.$$

(b) Find the probability that 2 of the $X_i$’s are in $[0, .2)$, 3 of the $X_i$’s are in $(.2, .7)$, and 5 of the $X_i$’s are in $[.7, 1]$. [5]

Solution: Let $Y_1$ be the number of $X_i$’s that are in $[0, .2)$, $Y_2$ the number of $X_i$’s that are in $(.2, .7)$, and $Y_3$ the number of $X_i$’s that are in $[.7, 1]$. Then $(Y_1, Y_2, Y_3)^T$ has a Multinomial distribution with parameters $n = 10$, $p_1 = .2$, $p_2 = .5$, and $p_3 = .3$, and we wish to compute the probability $P(Y_1 = 2, Y_2 = 3, Y_3 = 5)$. This is computed as

$$P(Y_1 = 2, Y_2 = 3, Y_3 = 5) = \frac{10!}{2!3!5!} .2^2 .5^3 .3^5$$

$$= 2520 \times .00001215 = 0.030618.$$
Formulas:

- The Uniform(0,1) distribution has pdf

\[ f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases} \]