Queen’s University  
Department of Mathematics and Statistics  

MTHE/STAT 353  
Midterm Examination    February 17, 2017  

• Total points = 30. Duration = 58 minutes. 

• This is a closed book exam. 

• One 8.5 by 11 inch sheet of notes, written on both sides, is permitted. 

• A simple calculator is permitted. 

• Write the answers in the space provided, continue on the backs of pages if needed. 

• SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks. 

• Marks per part question are shown in brackets at the right margin. 

• The last page contains formulas you may find useful. Please check this page first. 

Marks: Please do not write in the space below. 

Problem 1 [10]  

Problem 2 [10]  

Problem 3 [10]  

Total: [30]  

1. (a) Let $X_1, X_2, X_3$ be independent $N(0, 1)$ random variables. Find the probability density function of $U = X_1^2 + X_2^2 + X_3^2$. [5]

(b) Suppose that the random vector $Y = (Y_1, Y_2, Y_3)$ is uniformly distributed on the sphere of radius 1 centred at the origin; that is, $Y$ has joint probability density function

$$f_Y(y_1, y_2, y_3) = \begin{cases} \frac{3}{4\pi} & \text{if } (y_1, y_2, y_3) \in S \\ 0 & \text{otherwise,} \end{cases}$$

where $S = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 \leq 1\}$ is the sphere of radius 1 centred at $(0, 0, 0)$. Let $V = Y_1^2 + Y_2^2 + Y_3^2$. Find the probability density function of $V$ and find $E[V]$. [5]
2. Let $X_1, \ldots, X_n$ be a sequence of independent random variables uniformly distributed on the interval $(0, 1)$, and let $X_{(1)}, \ldots, X_{(n)}$ denote their order statistics. For fixed $k$ let $g_n(x)$ denote the probability density function of $nX_{(k)}$. Find $g_n(x)$ and show that

$$
\lim_{n \to \infty} g_n(x) = \begin{cases} 
x^{k-1} / (k-1)! e^{-x} & \text{for } x \geq 0 \\
0 & \text{for } x < 0
\end{cases}
$$

which is the Gamma$(k,1)$ density.
3. Let $X_1, \ldots, X_6$ be independent random variables uniformly distributed on the interval $(0, 1)$. Find the pdf of $U = \min\{\max(X_1, X_2), \max(X_3, X_4), \max(X_5, X_6)\}$. [10]
Formulas:

- The volume of a sphere of radius $r$ is $\frac{4}{3} \pi r^3$.

- Uniform distribution on the interval $(0, 1)$ has pdf

$$f(x) = \begin{cases} 
1 & \text{if } 0 < x < 1 \\
0 & \text{otherwise}
\end{cases}$$

- The standard normal distribution has pdf

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{for } x \in \mathbb{R}$$

- Beta density with parameters $\alpha$ and $\beta$:

$$f(x) = \begin{cases} 
\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{if } 0 < x < 1 \\
0 & \text{otherwise}
\end{cases}$$

- Gamma density with parameters $r$ and $\lambda$:

$$f(x) = \begin{cases} 
\frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} & \text{if } x > 0 \\
0 & \text{otherwise}
\end{cases}$$