1. Consider the simple random walk, \( \{X_n : n \geq 0\} \), starting at 0 (\( X_0 = 0 \)), where the probability of moving up at each step is \( p \) and the probability of moving down at each step is \( q = 1 - p \). Let \( M_n \) denote the maximum value attained by the walk up to time \( n \): \( M_n = \max_{0 \leq i \leq n} X_i \). For \( r \geq 1 \), show that
\[
P(M_n \geq r, X_n = b) = \begin{cases} P(X_n = b) & \text{if } b \geq r \\ (q/p)^{r-b}P(X_n = 2r - b) & \text{if } b < r. \end{cases}
\]
(Hint: For \( b < r \), there is a correspondence between sample paths of length \( n \) making up the event \( \{M_n \geq r, X_n = b\} \) and those making up the event \( \{X_n = 2r - b\} \). Find the correspondence and use it to relate the probabilities of these two events.)

2. Consider the Galton-Watson branching process \( \{X_n : n \geq 0\} \) starting with one individual (\( X_0 = 1 \)) and Geometric(1/2) family size distribution on the nonnegative integers; i.e.,
\[
P(\text{family size is } j) = \left(\frac{1}{2}\right)^{j+1} \quad \text{for } j = 0, 1, 2, \ldots
\]
Let \( Y \equiv \min\{n : X_n = 0\} \) denote the first generation to have no individuals. Find the probability mass function of \( Y \). Hint: Let \( F \) denote the cumulative distribution function of \( Y \). Then \( F(1) = 1/2 \); find \( F(k) \), for \( k \geq 2 \), by conditioning on the size of the first generation. If the first generation has \( j \) individuals then each of those \( j \) individuals is the root of an independent branch whose “lifetime” has the same distribution as that of \( Y \), and \( Y \) is less than or equal to \( k \) if and only if the maximum of the lifetimes of those \( j \) independent branches is less than or equal to \( k - 1 \). Obtain a recursion for \( F(k) \) and show by induction that \( F(k) = k/(k + 1) \) for \( k \geq 1 \).

3. Consider the Galton-Watson branching process as discussed in class. Assume that the family size distribution puts probability 0 on 0 offspring, so the population will never die out. Now pick two individuals with replacement from generation \( n > 1 \). Let \( L \) denote the index of the generation that contains their most recent common ancestor. For \( r = 0, \ldots, n - 2 \), show that
\[
P(L < r) = 1 - E[X_r^{-1}].
\]
Hint: For \( r < n \) condition on \( X_r = k \) and use a symmetry argument to show that the probability that two individuals chosen at random with replacement in generation \( n \) have the same generation \( r \) ancestor is \( 1/k \).
4. The infinite binary tree is a tree graph with one root node which has two child nodes, and each child node has a further two child nodes, ad infinitum. Consider a particle which performs a random walk on the infinite binary tree, starting at the root node at time 0. From the root node the particle moves to one of its child nodes, each with probability $1/2$. Whenever the particle is at a non-root node, it next moves to one of that node’s child nodes, each with probability $1/3$, or to that node’s parent node with probability $1/3$. Label the child nodes of the root node as $L$ (for left) and $R$ (for right). Compute the probability that the particle ever visits the $R$ child node. Hint: To start condition on the first move of the particle. For one of the resulting conditional probabilities condition again, this time on the whether or not the particle ever returns to the root node.

5. Suppose a random graph is constructed in the following manner. There are $n$ nodes. Each node chooses a node other than itself at random and an edge is placed between that node and the chosen node. Find the probability that a 5-node graph constructed in this way is connected. Hint: Start by constructing the supernode as in class, let $N$ denote the size of the supernode, and condition on $N$. After this the solution differs from the example in class but with only 5 nodes you can work out each of the conditional probabilities separately.

*6. Polya’s urn model supposes that an urn initially contains $r$ red and $b$ blue balls. At each stage a ball is randomly selected from the urn and is then returned along with $m$ other balls of the same color. Let $X_k$ be the total number of red balls drawn in the first $k$ selections, and let

$$Y_i = \begin{cases} 1 & \text{if the } i\text{th draw is red} \\ 0 & \text{if the } i\text{th draw is blue.} \end{cases}$$

(a) Show by induction and conditioning that $E[Y_k]$ is the same for all $k \geq 1$, and give this common value.

(b) From part(a) one can easily compute $E[X_k]$. What is the value of $E[X_k]$? Now give an alternative intuitive argument for the value of $E[X_k]$ via the following symmetry. Number the initial $r$ red and $b$ blue balls, so the urn contains one type $i$ red ball, for each $i = 1,\ldots,r$; as well as one type $j$ blue ball, for each $j = 1,\ldots,b$. Now suppose that whenever a red ball is chosen it is returned along with $m$ others of the same type, and similarly whenever a blue ball is chosen it is returned along with $m$ others of the same type. By symmetry each ball type should evolve statistically identically.