1. **From Sheet.** (5 marks)

If \( b \geq r \) then \( \{X_n = b\} \Rightarrow \{M_n \geq r\} \). Therefore, it is clear that

\[
P(M_n \geq r, X_n = b) = P(X_n = b) \quad \text{if} \quad b \geq r.
\]

If \( b < r \), consider a sample path satisfying \( \{M_n \geq r, X_n = b\} \). Let \( t_r \) denote the first time the path reaches \( r \). Reflect the portion of the sample path from \( t_r \) to \( n \) about the line \( y = r \). The resulting sample path goes from 0 at time 0 to \( 2r - b \) at time \( n \).

This operation defines a one-to-one correspondence between the following two sets of sample paths:

1. those that reach state \( r \) by time \( n \) and are in state \( b \) at time \( n \) (i.e., those that make up the event \( \{M_n \geq r, X_n = b\} \));
2. those that are in state \( 2r - b \) at time \( n \) (i.e., those that make up the event \( \{X_n = 2r - b\} \)).

Therefore, if \( n_1 \) is the number of sample paths in set (1) and \( n_2 \) is the number of sample paths in set (2), then \( n_1 = n_2 \). Also, each sample path in set (1) has probability \( p^{(n+b)/2} q^{(n-b)/2} \) (each such path must have \( (n + b)/2 \) jumps up and \( (n - b)/2 \) jumps down). Similarly, each sample path in set (2) has probability \( p^{(n+2r-b)/2} q^{(n-2r+b)/2} \). Therefore, we have

\[
P(M_n \geq r, X_n = b) = n_1 p^{(n+b)/2} q^{(n-b)/2} = n_2 p^{(n+b)/2} q^{(n-b)/2} = n_2 p^{b-r} p^{(n+2r-b)/2} q^{r-b} q^{(n-2r+b)/2} = (q/p)^{r-b} P(X_n = 2r - b).
\]

2. **From Sheet.** (7 marks) As suggested in the hint, let \( F \) denote the cumulative distribution function of \( Y \), the “lifetime” of the population. Since \( Y \geq 1 \),

\[
F(1) = P(Y \leq 1) = P(Y = 1) = P(\text{family size is 0}) = \frac{1}{2}.
\]
where the last equality follows since the first generation is empty if and only if the root individual from generation 0 has exactly 0 offspring. Now consider \( F(k) \) for \( k \geq 2 \). Conditioning on the size of the first generation we have

\[
F(k) = P(Y \leq k) = \sum_{j=0}^{\infty} P(Y \leq k \mid X_1 = j) P(X_1 = j)
\]

\[
= \sum_{j=0}^{\infty} P(Y \leq k \mid X_1 = j) \left( \frac{1}{2} \right)^{j+1}.
\]

If we let \( Y_i \) denote the “lifetime” of the branch rooted at the \( i \)th individual of generation 1 then as noted in the hint, given that \( X_1 = j \), \( Y_1, \ldots, Y_j \) are independent each with the same distribution as that of \( Y \). Now \( P(Y \leq k \mid X_1 = 0) = 1 \) and for \( j > 0 \), \( P(Y \leq k \mid X_1 = j) = P(\max(Y_1, \ldots, Y_j) \leq k - 1) \). Therefore,

\[
F(k) = \sum_{j=0}^{\infty} P(Y \leq k \mid X_1 = j) \left( \frac{1}{2} \right)^{j+1}
\]

\[
= \frac{1}{2} + \sum_{j=1}^{\infty} P(\max(Y_1, \ldots, Y_j) \leq k - 1) \left( \frac{1}{2} \right)^{j+1}
\]

\[
= \frac{1}{2} + \sum_{j=1}^{\infty} P(Y_1 \leq k - 1)^j \left( \frac{1}{2} \right)^{j+1}
\]

\[
= \frac{1}{2} + \sum_{j=1}^{\infty} F(k - 1)^j \left( \frac{1}{2} \right)^{j+1}
\]

\[
= \sum_{j=0}^{\infty} F(k - 1)^j \left( \frac{1}{2} \right)^{j+1}
\]

\[
= \frac{1}{2} \frac{1}{1 - F(k - 1)(1/2)}
\]

\[
= \frac{1}{2 - F(k - 1)}.
\]

We now show by induction that \( F(k) = k/(k + 1) \) solves the above recursion. Since \( F(1) = 1/2 \) the statement is true for \( k = 1 \). Now assume the statement is true for \( k - 1 \). Then

\[
F(k) = \frac{1}{2 - F(k - 1)} = \frac{1}{2 - (k-1)/k} = \frac{k}{2k - k + 1} = \frac{k}{k + 1}.
\]

Therefore, the statement is true for \( k \). The probability mass function of \( Y \) is now obtained as

\[
p_Y(k) = P(Y = k) = F(k) - F(k - 1) = \frac{k}{k + 1} - \frac{k - 1}{k} = \frac{1}{k(k + 1)},
\]

which is valid for \( k = 1, 2, \ldots \), and \( p_Y(x) = 0 \) otherwise.
3. From Sheet. (5 marks) Fix generations \( r \) and \( n, r < n \), and suppose that \( X_r = k \). Each individual in generation \( n \) has exactly one ancestor in generation \( r \). By symmetry (since all individuals have the same family size distribution), each of the \( k \) individuals from generation \( r \) is equally likely to have been the ancestor of a randomly chosen individual from generation \( n \). Therefore, when we randomly choose two individuals from generation \( n \) (with replacement) the probability that individual \( i \) in generation \( r \) is a common ancestor is \( \frac{1}{k} \). Since any of the \( k \) individuals in generation \( r \) could be a common ancestor, the probability that the two randomly chosen individuals from generation \( n \) have a common ancestor in generation \( r \) is \( \frac{1}{k} \). On the other hand, the generation of the most recent ancestor of these two individuals is less than \( r \) if and only if they do not have a common ancestor in generation \( r \). We have argued that

\[
P(L < r \mid X_r = k) = 1 - \frac{1}{k}.
\]

Therefore, by unconditioning we obtain

\[
P(L < r) = \sum_k \left( 1 - \frac{1}{k} \right) P(X_r = k) = 1 - E[X_r^{-1}].
\]

4. From Sheet. (5 marks) Let \( X_n \) denote the location of the particle after the \( n \)th move. Call the root node state 0, the left child of the root state \( L \), and the right child of the root state \( R \). Let \( A \) denote the event that the particle ever visits state \( R \). We wish to compute the probability \( P(A \mid X_0 = 0) \). Conditioning on the first step of the particle we have

\[
P(A \mid X_0 = 0) = P(A \mid X_1 = R, X_0 = 0)P(X_1 = R \mid X_0 = 0)
+ P(A \mid X_1 = L, X_0 = 0)P(X_1 = L \mid X_0 = 0)
\]

\[
= \frac{1}{2} + \frac{1}{2}P(A \mid X_1 = L, X_0 = 0) \tag{1}
\]

since the particle is equally likely to go right or left on the first move and if the particle moves right on the first move then the event \( A \) occurs with (conditional) probability 1. Now we consider \( P(A \mid X_1 = L, X_0 = 0) = P(A \mid X_0 = L) \), where the latter expression reflects the fact that we can assume the particle now starts in state \( L \) (it is easy to see that this process is temporally homogeneous). Let \( B \) denote the event that the particle ever visits state 0. Conditioning on the event \( B \) we have

\[
P(A \mid X_0 = L) = P(A \mid B, X_0 = L)P(B \mid X_0 = L)
+ P(A \mid B^c, X_0 = L)P(B^c \mid X_0 = L).
\]

Now \( P(A \mid B^c, X_0 = L) = 0 \) and \( P(B \mid X_0 = L) = 1/2 \) (the latter probability is from an example in class in which we computed the probability that a simple random walk starting in state 0 ever reaches state 1 to be \( p/q \) if \( p < q \); here \( p = 1/3 \) and
\( q = 2/3 \). Also, given the particle does make it back to state 0 we are back to the starting point, giving \( P(A \mid B, X_0 = L) = P(A \mid X_0 = 0) \). Thus, we now have that \( P(A \mid X_1 = L, X_0 = 0) = (1/2)P(A \mid X_0 = 0) \). Plugging this back into Eq.(1) we have

\[
P(A \mid X_0 = 0) = \frac{1}{2} + \frac{1}{4}P(A \mid X_0 = 0),
\]

or

\[
P(A \mid X_0 = 0) = \left(\frac{1}{2}\right)\left(\frac{4}{3}\right) = \frac{2}{3}.
\]

5. From Sheet. (5 marks) Let \( N \) denote the size of the supernode as in the example done in class (or Section 3.6.2 of the text). For \( n = 5 \), and since a node does not choose itself, the possible values of \( N \) are 2, 3, 4 or 5. First, note that if \( N = 4 \) or \( N = 5 \) the graph will be connected with probability 1. If \( N = 3 \) then the graph will be connected if and only if at least one of the two remaining nodes connects to the supernode. That is,

\[
P(\text{connected} \mid N = 3) = P(\text{at least one of the two remaining nodes is connected to the supernode})
\]

\[
= 1 - P(\text{neither of the two remaining nodes is connected to the supernode})
\]

\[
= 1 - \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{15}{16}.
\]

If \( N = 2 \), then we may further condition on how many of the three remaining nodes has an edge to the supernode. Let \( X \) denote the number of nodes not in the supernode but with an edge to the supernode. Given \( N = 2 \), if \( X = 2 \) or \( X = 3 \) then the graph is connected with probability 1. If \( X = 0 \), then the graph is not connected. If \( X = 1 \), then the graph is connected if at least one of the 2 nodes that do not have an edge to the supernode has an edge to the node that does have an edge to the supernode. Since neither of these two nodes chooses the supernode, each will choose the node with the edge to the supernode with probability \( 1/2 \), and the probability that at least one of them has an edge to the node with an edge to the supernode is \( 1 - (1/2)(1/2) = 3/4 \). Given \( N = 2 \), \( X \) has a Binomial(3, 1/2) distribution. Thus, we have

\[
P(\text{connected} \mid N = 2) = \sum_{k=0}^{3} P(\text{connected} \mid N = 2, X = k)P(X = k \mid N = 2)
\]

\[
= (0)P(X = 0 \mid N = 2) + \frac{3}{4}P(X = 1 \mid N = 2)
\]

\[
+ (1)P(X = 2 \mid N = 2) + (1)P(X = 3 \mid N = 2)
\]

\[
= \left(\frac{1}{2}\right)^3 \left[\frac{3}{4}\left(\begin{array}{c} 3 \\ 1 \end{array}\right) + \left(\begin{array}{c} 3 \\ 2 \end{array}\right) + \left(\begin{array}{c} 3 \\ 3 \end{array}\right)\right] = \frac{25}{32}.
\]
The distribution of $N$ is given by

\[
\begin{align*}
P(N = 2) &= \frac{1}{4} \\
P(N = 3) &= \frac{3}{4} \times \frac{1}{2} \\
P(N = 4) &= \frac{3}{4} \times \frac{1}{2} \times \frac{3}{4} = \frac{9}{32} \\
P(N = 5) &= \frac{3}{4} \times \frac{1}{2} \times \frac{1}{4} = \frac{3}{32}.
\end{align*}
\]

Therefore,

\[
P(\text{connected}) = P(\text{connected} \mid N = 2)P(N = 2) + P(\text{connected} \mid N = 3)P(N = 3) \\
+ P(\text{connected} \mid N = 4)P(N = 4) + P(\text{connected} \mid N = 5)P(N = 5)
\]

\[
= \frac{25}{32} \times \frac{1}{4} + \frac{15}{16} \times \frac{3}{8} + 1 \times \frac{9}{32} + 1 \times \frac{3}{32} = \frac{59}{64} \approx 0.9219.
\]

6. From Sheet. (8 marks)

(a) (5 marks) Since it is clear that $E[Y_1] = r/(r + b)$ we wish to show that $E[Y_k] = r/(r + b)$ for all $k \geq 1$. We will use an induction argument, so assume the pattern holds for $i = 1, \ldots, k - 1$. After $k - 1$ draws, given $X_{k-1} = j$ the urn contains $r + mj$ red balls and a total of $r + b + (k - 1)m$ balls. Therefore, $E[Y_k \mid X_{k-1} = j] = \frac{r + mj}{r + b + (k-1)m}$, so conditioning on $X_{k-1}$ we have

\[
E[Y_k] = \sum_{j=0}^{k-1} E[Y_k \mid X_{k-1} = j]P(X_{k-1} = j)
\]

\[
= \sum_{j=0}^{k-1} \frac{r + mj}{r + b + (k-1)m}P(X_{k-1} = j)
\]

\[
= \frac{r + mE[X_{k-1}]}{r + b + (k-1)m}
\]

\[
= \frac{r + m(E[Y_1] + \ldots + E[Y_{k-1}])}{r + b + (k-1)m}
\]

\[
= \frac{r + m(k-1)r/(r + b)}{r + b + (k-1)m}
\]

\[
= \frac{r}{r + b} \left[ \frac{1 + m(k-1)/(r + b)}{1 + m(k-1)/(r + b)} \right] = \frac{r}{r + b},
\]

where the fifth equality follows from the induction hypothesis.

(b) (3 marks) From $X_k = Y_1 + \ldots + Y_k$ and part(a) the expected value of $X_k$ is $E[X_k] = kr/(r + b)$. We now give an intuitive argument for this value of $E[X_k]$. We define
the types as suggested in the hint. The symmetry argument is that each type should evolve statistically identically so that the expected number of type $i$ balls in the urn after $k$ draws is the same as the expected number of type $j$ balls in the urn after $k$ draws, for any $i \neq j$ and for all $k \geq 1$. Since after $k$ draws there are exactly $r + b + km$ balls in the urn and there are $r + b$ types the expected number of type $i$ balls should be

$$\frac{r + b + km}{r + b} = 1 + \frac{km}{r + b}.$$ 

Since $r$ of the types are red the expected number of red balls in the urn after $k$ draws should be $r + \frac{rkm}{r+b}$. But this represents $\frac{k}{r+b}$ red ball draws.