1. In the Galton-Watson branching process starting with a single individual \((X_0 = 1)\), find the probability of extinction

(a) when the family size distribution is Geometric with parameter \(p < 1/2\); i.e.,

\[
P(Z = k) = pq^k \quad \text{for } k = 0, 1, \ldots,
\]

where \(Z\) is a typical family size; and

(b) when the family size is 0, 1, or 2 with probabilities 1/4, 1/4, and 1/2, respectively.

2. Let \(\{X_n : n \geq 0\}\) be a time-homogeneous Markov chain. Show that, for \(1 < r < n\) and states \(k\) and \(x_i, i = 1, \ldots, r - 1, r + 1, \ldots, n\),

\[
P(X_r = k \mid X_i = x_i \text{ for } i = 1, \ldots, r - 1, r + 1, \ldots, n) = P(X_r = k \mid X_{r-1} = x_{r-1}, X_{r+1} = x_{r+1}).
\]

3. A random sequence of polygons is generated by picking two edges of the current polygon at random (without replacement), joining their midpoints, and picking one of the resulting smaller polygons at random to be the next in the sequence. Let \(X_n + 3\) be the number of edges of the \(n\)th polygon thus constructed (e.g., if \(X_n = 0\) then the \(n\)th polygon is a triangle). Give the state space and the transition probability matrix of the Markov chain \(\{X_n : n \geq 0\}\).

4. Let \(\{X_n : n \geq 0\}\) be a time-homogeneous Markov chain with state space \(S = \{1, 2, 3, 4, 5, 6\}\) and the transition probability matrix is

\[
P = \begin{bmatrix}
\frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}.
\]

(a) Classify each state according to whether it is transient or recurrent and give its period. Identify the equivalence classes of \(S\).
(b) For each recurrent state \( i \), compute

\[
f_{ii}(n) = P(\text{first return to state } i \text{ is at time } n \mid X_0 = i)
\]

and compute \( \mu_i \), the mean time to return to state \( i \), starting in state \( i \).

5. Consider the simple symmetric random walk in two dimensions on the set of points

\( S = \{ (x, y) : x, y = 0, \pm 1, \pm 2, \ldots \} \): this process is a sequence \( \{X_n : n \geq 0\} \) of random vectors such that \( X_{n+1} = X_n + Y_{n+1} \), where \( Y_1, Y_2, \ldots \) are independent and identically distributed random vectors, each with distribution given by

\[
P(Y_n = y) = \begin{cases} 
1/4 & \text{for } y = (1, 0) \\
1/4 & \text{for } y = (0, -1) \\
1/4 & \text{for } y = (-1, 0) \\
1/4 & \text{for } y = (0, 1)
\end{cases}
\]

Show that this Markov chain is recurrent (i.e., all states are recurrent). \textit{Hint:} Just focus on state \((0, 0)\) since the Markov chain is irreducible; compute \( P(X_{2n} = (0, 0) \mid X_0 = (0, 0)) \) and use the identity \( \sum_{n=0}^{\infty} \binom{n}{m}^2 = \binom{2n}{n} \) and Stirling’s approximation.

\[\star\] 6. Consider a Markov chain on the set \( S = \{0, 1, 2, \ldots\} \) with transition probabilities \( p_{i,i+1} = a_i, i \geq 0, p_{i,0} = 1 - a_i \), where \( \{a_i, i \geq 0\} \) is a sequence of constants which satisfy \( 0 < a_i < 1 \) for all \( i \). Let \( b_0 = 1, b_i = a_0 a_1 \cdots a_{i-1} \) for \( i \geq 1 \). Show that the chain is recurrent if and only if \( b_i \to 0 \) as \( i \to \infty \) and show that the chain is positive recurrent if and only if \( \sum_{i=0}^{\infty} b_i < \infty \). (Hint: First show that the chain is irreducible and then focus only on state 0. Let \( T_0 \) be the first time the chain returns to state 0, starting in state 0. The chain is recurrent if and only if \( P(T_0 < \infty) = 1 \). To compute the mean recurrence time, use \( E[T_0] = \sum_{n=0}^{\infty} P(T_0 > n) \)).