Stochastic Processes

Assignment #2
Due Wednesday, Nov.4

Starred questions are for 855 students only.

1. In the Galton-Watson branching process starting with a single individual \(X_0 = 1\), find the probability of extinction

(a) when the family size distribution is Geometric with parameter \(p < 1/2\); i.e.,

\[
P(Z = k) = pq^k \quad \text{for } k = 0, 1, \ldots ,
\]

where \(Z\) is a typical family size; and

(b) when the family size is 0, 1, or 2 with probabilities 1/4, 1/4, and 1/2, respectively.

2. Suppose \(S = \{1, 2, 3, 4, 5, 6\}\) and the transition probability matrix is

\[
P = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\
\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}.
\]

(a) Classify each state according to whether it is transient or recurrent and give its period. Identify the equivalence classes of \(S\).

(b) For each recurrent state \(i\), compute

\[
f_{ii}(n) = P(\text{first return to state } i \text{ is at time } n \mid X_0 = i)
\]

and compute \(\mu_i\), the mean time to return to state \(i\), starting in state \(i\).

3. Consider the simple random walk in two dimensions on the set of points \(S = \{(x, y) : x, y = 0, \pm 1, \pm 2, \ldots \}\): this process is a sequence \(\{X_n : n \geq 0\}\) of random vectors such that \(X_{n+1} = X_n + Y_{n+1}\), where \(Y_1, Y_2, \ldots\) are independent and identically distributed random vectors, each with distribution given by

\[
P(Y_n = y) = \begin{cases}
p_1 & \text{for } y = (1, 0) \\
p_2 & \text{for } y = (0, -1) \\
p_3 & \text{for } y = (-1, 0) \\
p_4 & \text{for } y = (0, 1)
\end{cases}
\]
where $p_1 + p_2 + p_3 + p_4 = 1$. Show that this Markov chain is recurrent (i.e., all states are recurrent) if $p_1 = p_2 = p_3 = p_4 = 1/4$. Show that it is transient if $p_1 \neq p_3$. \textit{Hint:} For the symmetric case, just focus on state $(0,0)$ since the Markov chain is irreducible; compute $P(X_{2n} = (0,0) \mid X_0 = (0,0))$ and use the identity $\sum_{m=0}^{n} \binom{n}{m}^2 = \binom{2n}{n}$ and Stirling’s approximation. For the nonsymmetric case compare to a one dimensional random walk.

4. A random sequence of polygons is generated by picking two edges of the current polygon at random (without replacement), joining their midpoints, and picking one of the resulting smaller polygons at random to be the next in the sequence. Let $X_n + 3$ be the number of edges of the $n$th polygon thus constructed (e.g., if $X_n = 0$ then the $n$th polygon is a triangle). Give the state space and the transition probability matrix of the Markov chain $\{X_n : n \geq 0\}$.

5. Consider an irreducible discrete time Markov chain with state space $S$ and period $d$, where $1 < d < \infty$. Let $i_0, \ldots, i_{d-1}$ be $d$ distinct states satisfying $p_{i_j, i_{j+1}} > 0$ for $j = 0, \ldots, d - 2$. That is, the sample path segment $i_0 \rightarrow i_1 \rightarrow \ldots \rightarrow i_{d-1}$ has positive probability ($d$ such distinct states must exist – why?). Define

$$C_j = \{i \in S : p_{i_j, i_{(nd)}} > 0 \text{ for some } n \geq 0\}.$$  

The sets $C_0, \ldots, C_{d-1}$ are called the \textit{cyclic classes} of the Markov chain.

(a) Show that $i \in C_j$ if and only if $p_{i_j, i_{(nd)}} > 0$ for some $n \geq 0$. \textit{Hint:} For example, for the forward part, show that if the conclusion does not hold then you can construct a path from $i_j$ back to $i_j$ whose number of steps is not a multiple of $d$.

(b) Show that $C_0, \ldots, C_{d-1}$ are disjoint. \textit{Hint:} Show by contradiction that a state $i$ cannot be in any two of these classes.

(c) Show that every state $i \in S$ belongs to at least one of $C_0, \ldots, C_{d-1}$. This, together with part(b), implies that $C_0, \ldots, C_{d-1}$ partition the state space $S$.

(d) Show that starting in class $C_j$ the process can revisit $C_j$ only at times that are a multiple of $d$. Hence, show that the Markov chain cycles through the classes $C_0, \ldots, C_{d-1}$, and show that it does so in that order; i.e., for each class $k = 0, \ldots, d - 1$ and for each $i \in C_k$,

$$\sum_{j \in C_{k+1}} p_{ij} = 1,$$

where $C_d \equiv C_0$.

*6. (a) Give an example of a Markov chain $\{X_n : n \geq 0\}$ and a Markov chain $\{Y_n : n \geq 0\}$ such that the stochastic process $\{Z_n : n \geq 0\}$, where $Z_n = X_n + Y_n$, is not a Markov chain (and show that it is not a Markov chain).
(b) Give an example of an array of real numbers \( \{a_i(n)\}, \ i \geq 0, \ n \geq 1, \) such that

\[
a = \lim_{n \to \infty} \sum_{i=0}^{\infty} a_i(n) \neq \sum_{i=0}^{\infty} \lim_{n \to \infty} a_i(n) = b,
\]

where both \( a \) and \( b \) are finite.