Stochastic Processes

Assignment #3
Due Monday, November 13

Starred questions are for 855 students only.

1. Let $X = \{X_n : n \geq 0\}$ be an irreducible, positive recurrent Markov chain with period $d > 1$, state space $S$, transition probability matrix $P$, stationary distribution $\pi$, and cyclic classes $C_0, \ldots, C_{d-1}$ (cf., Problem 5 on homework 2). Consider the transition matrix $Q = P^d$. Let $Y = \{Y_n : n \geq 0\}$ be a Markov chain with transition matrix $Q$.

(a) Show that the $Y$ chain has $d$ equivalence classes, namely $C_0, \ldots, C_{d-1}$.

(b) Show that all states in the $Y$ chain have period 1. $Hint$: One way to do this is to show that if state $i$ has period $d_0 > 1$, then this leads to a contradiction.

(c) Based on the interpretation of $\pi_i$ is the long run proportion of time that the $X$ chain spends in state $i$, and using parts (a) and (b), explain why

$$\sum_{i \in C_j} \pi_i = \frac{1}{d},$$

for $j = 0, \ldots, d - 1$, and why $p_{ii}(nd) \to d\pi_i$ as $n \to \infty$.

2. A random sequence of polygons is generated by picking two edges of the current polygon at random (without replacement), joining their midpoints, and picking one of the resulting smaller polygons at random to be the next in the sequence. Let $X_n + 3$ be the number of edges of the $n$th polygon thus constructed (e.g., if $X_n = 0$ then the $n$th polygon is a triangle). The process $\{X_n : n \geq 0\}$ is an irreducible, positive recurrent Markov chain. Write down the state space and transition matrix of this Markov chain and find the stationary distribution.

3. Let $\{X_n : n \geq 0\}$ be an irreducible, aperiodic, positive recurrent Markov chain, with stationary distribution $\pi$.

(a) Show that $P(X_n = j) \to 1/\mu_j$ as $n \to \infty$, where $\mu_j$ is the mean return time to state $j$. 
(b) Show that if \( \{x_n\}_{n=1}^{\infty} \) is a sequence of real numbers satisfying \( x_n \to x \) as \( n \to \infty \) for some limit \( x \in (-\infty, \infty) \), then \( n^{-1} \sum_{i=1}^{n} x_i \to x \). Hence show that
\[
\pi_j = \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} P(X_m = j)
\]

(we often interpret \( \pi_j \) to be the long run proportion of time that the chain is in state \( j \)). The terms \( n^{-1} \sum_{i=1}^{n} x_i \) are called Cesàro means and the result is a basic result about these means, if you want to look it up.

4. A chess piece performs a symmetric random walk on a chessboard if at each step it is equally likely to make any one of its available moves. Suppose a rook and a bishop perform independent random walks on a 4 \times 4 chessboard. If they start together at a corner, show that the expected number of steps until they meet again at the same corner is \( 448/3 \). (Hint: Formulate as a random walk on a graph with 16 \times 8 nodes, compute the degree of each node, and use the results of the random walk on a graph example in class).

5. Let \((X, Y)\) be a discrete bivariate random vector with sample space \( S_X \times S_Y \) and joint probability mass function \( p(x, y) \), where \( S_X \) and \( S_Y \) are discrete subsets of the real line. That is, \( p(x, y) = P(X = x, Y = y) > 0 \) for \( x \in S_X \) and \( y \in S_Y \), and \( p(x, y) = 0 \) otherwise. Suppose we generate (on a computer, say) a random sequence of 2-tuples in the following way. We start with an arbitrary 2-tuple, \((x_0, y_0)\) \( \in S_X \times S_Y \). If our current 2-tuple is \((x_n, y_n)\), to get the next 2-tuple in the sequence we first choose one of the co-ordinates at random (either \( x_n \) or \( y_n \)). If we choose the first co-ordinate then we take a draw from the conditional distribution of \( Y \) given \( X = x_n \). Letting \( z \) denote the value of this draw, the next 2-tuple in our sequence is \((x_n+1, y_n+1) = (x_n, z)\). If we choose the second co-ordinate then we take a draw from the conditional distribution of \( X \) given \( Y = y_n \). Letting \( z \) denote the value of this draw, the next 2-tuple in our sequence is \((x_n+1, y_n+1) = (z, y_n)\). The sequence of 2-tuples generated in this way is a random sequence. Let us denote this sequence by \( \{(X_n, Y_n) : n \geq 0\} \), where \((X_0, Y_0) = (x_0, y_0)\). The fact that the sequence \( \{(X_n, Y_n) : n \geq 0\} \) has the Markov property is implicit in the sequence definition, which tells us the distribution from which to draw the next state given the current state, and this distribution is independent of any past states.

(a) Show that the Markov chain \( \{(X_n, Y_n) : n \geq 0\} \) is irreducible.
(b) Show that the Markov chain \( \{(X_n, Y_n) : n \geq 0\} \) is aperiodic.
(c) Show that for any \((x, y) \in S_X \times S_Y\),
\[
\lim_{n \to \infty} P((X_n, Y_n) = (x, y) \mid (X_0, Y_0) = (x_0, y_0)) = p(x, y).
\]
⋆6. Convergence Rates. Let \( X = \{X_n : n \geq 0\} \) be a Markov chain with finite state space \( S \) and transition probabilities \( p_{ij} \) such that \( p_{ij} > 0 \) for all \( i, j \in S \). Show that there exists \( \lambda \in (0, 1) \) such that
\[
|p_{ij}(n) - \pi_j| < \lambda^n,
\]
where \( \pi \) is the stationary distribution.

Hint: Let \( Y \) be a Markov chain independent of \( X \) with initial distribution \( \pi \). Use the lecture notes from class to show that
\[
|p_{ij}(n) - \pi_j| \leq P(T \geq n \mid X_0 = i),
\]
where \( T = \min\{n \geq 1 : (X_n, Y_n) = (s, s)\} \), the first time the two chains \( X \) and \( Y \) couple at state \((s, s)\). Then argue that we can write
\[
P(T \geq n \mid X_0 = i) = P(T \geq n \mid T \geq n - 1, X_0 = i)P(T \geq n - 1 \mid X_0 = i).
\]
Then recurse down. Then upper bound \( P(T \geq r + 1 \mid T \geq r, X_0 = i) \) for any \( r \).