Stochastic Processes

Assignment #3
Due Tuesday, Nov.4

Starred questions are for 855 students only.

1. Consider the discrete time GI/D/1 queue, described as follows. The system consists of a single server and a queue with unlimited capacity. Time is discretized and customers arrive only at the beginning of a time period. Let $Z_n$ be the number of customers arriving for service at the beginning of the $n$th time period and assume $Z_1, Z_2, \ldots$ are i.i.d. with common p.m.f. $f(k)$ on $\{0, 1, 2, \ldots\}$ and mean $\mu < 1$. A customer arriving for service begins being served immediately if the server is free and the queue is empty upon arrival; otherwise, the customer joins the end of the queue. If two or more customers arrive at the same time, then one of them will go into service immediately if the server is free and the queue is empty, while the other customers will go to the end of the queue. The customers are served in a first-come, first-served manner, so any customers in the queue will always be served before any newly arriving customers. In each time period the server begins serving (and completes the service of) exactly one customer if there are any to be served (from the queue if it is nonempty or a newly arriving customer if the queue is empty); otherwise, the server is idle for that time period. Let $X_n$ be the number of customers in the system (either in the queue or in service) at the start of the $n$th time period ($after$ any customers arriving at the start of the $n$th time period have arrived). Then $X = \{X_n : n \geq 0\}$ is a Markov chain with state space $S = \{0, 1, 2, \ldots\}$. Assume that a stationary distribution $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$ exists (it does) and let $G(s)$ be its probability generating function:

$$G(s) = \sum_{n=0}^{\infty} s^n \pi_n.$$

Write down the transition probabilities and show that

$$G(s) = (1 - \mu) \frac{(s - 1)G_Z(s)}{s - G_Z(s)},$$

where $G_Z(s)$ is the probability generating function of $Z_1$. In this problem finding $G(s)$ is about the best way to get at $\pi$ without knowing more about the distribution of $Z_1$. (Hint: The chain $\{X_n : n \geq 0\}$ follows the recursion $X_{n+1} = X_n - 1 + Z_{n+1}$ if $X_n > 0$ and $X_{n+1} = Z_{n+1}$ if $X_n = 0$. The transition probabilities, in terms of the $f(k)$, may
be easily deduced from these relationships. Write down the Global Balance Equations and convert these to a single equation for \( G(s) \) This equation will involve \( G_Z(s) \) — you will need to recall the generating function of a convolution. Solving for \( G(s) \) in terms of \( G_Z(s) \), you will still have an unknown value of \( \pi_0 \) in the expression. You can determine \( \pi_0 \) by using the boundary conditions \( \lim_{s \to 1} G(s) = 1 \) and \( \lim_{s \to 1} G_Z(s) = 1 \) together with L’Hospital’s rule, and using the fact that \( G_Z'(1) = \mu \).

2. Let \( \{X_n : n \geq 0\} \) be a recurrent, irreducible Markov chain with state space \( S \) and transition probability matrix \( P \), and let \( u \) be a positive solution of the equation \( u = uP \) (\( u \) is a row vector of dimension \(|S|\) with all positive entries).

(a) For \( i, j \in S \) and \( n \geq 1 \) define

\[ q_{ij}(n) = \frac{u_j}{u_i} p_{ji}(n), \]

where \( p_{ji}(n) \) is the \( n \)-step transition probability from \( j \) to \( i \) in the \( X \) chain. Show that \( q_{ij}(n) \) defines the \( n \)-step transition probabilities of some (not necessarily the same) recurrent, irreducible Markov chain with state space \( S \). \textit{Hint:} First show that the \( q_{ij}(1) \) give valid one-step transition probabilities for a Markov chain, say \( \{Y_n : n \geq 0\} \). Then show by induction that \( q_{ij}(n) \) give the \( n \)-step transition probabilities for this Markov chain. Then show that all states in this Markov chain communicate. Finally, show that this Markov chain is recurrent.

(b) Let \( \{Y_n : n \geq 0\} \) be the Markov chain with transition probabilities \( q_{ij}(1) \), as given in part(a). For \( i \neq j \) and \( n \geq 1 \), let

\[ g_{ij}(n) = P(Y_n = j, Y_{n-1} \neq j, \ldots, Y_1 \neq j \mid Y_0 = i) \]

be the probability that the first time the \( Y \) chain reaches state \( j \) is at time \( n \), given that it starts in state \( i \). Show that

\[ g_{ij}(n) = \frac{u_j}{u_i} l_{ji}(n) \]  \hspace{1cm} (1)

where \( l_{ji}(n) = P(X_n = i, X_{n-1} \neq j, \ldots, X_1 \neq j \mid X_0 = j) \) is the probability that the \( X \) chain is in state \( i \) at time \( n \) and at time \( n \) the chain is still completing a sojourn from state \( j \) back to state \( j \). \textit{Hint:} Show it for \( n = 1 \) and then use induction on \( n \) to show in general. For the induction step condition on \( X_1 \) and use the Markov property and time homogeneity a couple of times.

(c) By summing both sides of Eq.(1) over \( n \) show that \( u \) is unique up to a multiplicative constant. \textit{Hint:} Explain why you get 1 on the left hand side. Relate what you get on the right hand side to a sojourn from state \( j \) back to state \( j \) in the \( X \) chain.

(d) Show that all states of the \( X \) chain are null recurrent if and only if \( \sum_{i \in S} u_i = \infty \).
3. Let $\{X_n : n \geq 0\}$ be an irreducible, aperiodic, positive recurrent Markov chain, with stationary distribution $\pi$.

(a) Show that $P(X_n = j) \to 1/\mu_j$ as $n \to \infty$, where $\mu_j$ is the mean return time to state $j$.

(b) Show that if $\{x_n\}_{n=1}^\infty$ is a sequence of real numbers satisfying $x_n \to x$ as $n \to \infty$ for some limit $x \in (-\infty, \infty)$, then $n^{-1} \sum_{i=1}^n x_i \to x$. Hence show that

$$
\pi_j = \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^n P(X_m = j)
$$

(we often interpret $\pi_j$ to be the long run proportion of time that the chain is in state $j$). The terms $n^{-1} \sum_{i=1}^n x_i$ are called Cesàro means and the result is a basic result about these means, if you want to look it up.

4. Show that an irreducible Markov chain with finite state space and transition matrix $P$ is time reversible if and only if $P = DS$ for some symmetric matrix $S$ and diagonal matrix $D$ with strictly positive diagonal entries. Hint: For the forward direction you can take $S = \Pi P$, where $\Pi$ is a diagonal matrix with $i$th entry $\pi_i$ (why is this $S$ diagonal?).

5. Let $X = \{X_n : n \geq 0\}$ be a time-reversible Markov chain on a state space $S$ with transition probabilities $p_{ij}$. Let $C$ be a non-empty subset of the state space $S$. Define the Markov chain $Y = \{Y_n : n \geq 0\}$ on $S$ by the transition probabilities

$$
q_{ij} = \begin{cases} 
\beta p_{ij} & \text{if } i \in C \text{ and } j \notin C \\
p_{ij} & \text{otherwise},
\end{cases}
$$

for $i \neq j$, where $\beta$ is a constant satisfying $0 < \beta < 1$, and $q_{ii} = 1 - \sum_{j \neq i} q_{ij}$. Show that the Markov chain $Y$ is time reversible and find its stationary distribution (in terms of the stationary distribution of $X$).

⋆6. Let $X = \{X_n : n \geq 0\}$ be a time-homogeneous Markov chain with state space $S$ and transition probabilities $p_{ij}$, and let $A$ be a subset of $S$. Define $T_A = \min\{n \geq 0 : X_n \in A\}$ be the first passage time to $A$, and let $\eta_j = P(T_A < \infty \mid X_0 = j)$ be the probability that the chain ever visits $A$ given that it started in state $j$. If $j \in A$ then clearly $\eta_j = 1$. Also, by conditioning on the first step of the chain, we can see that the $\eta_j$ satisfy the equations

$$
\eta_j = \begin{cases} 
1 & \text{if } j \in A \\
\sum_{k \in S} p_{jk} \eta_k & \text{if } j \notin A.
\end{cases}
$$

Show that $(\eta_j)_{j \in S}$ is the minimal non-negative solution to these equations. That is, if $(x_j)_{j \in S}$ is any other non-negative solution, then $x_j \geq \eta_j$ for all $j \in S$. (Hint: show by induction that $x_j \geq P(T_A \leq n \mid X_0 = j)$ for all $n$).