Stochastic Processes

Assignment #3, Solutions

Total Marks: 39 for 455 and 47 for 855.

1. From Sheet. (13 marks)

(a) (4 marks) For \( n = 1 \), \( q_{ij}(1) \) is clearly nonnegative and for each \( i \in S \),
\[
\sum_{j \in S} q_{ij}(1) = \frac{1}{u_i} \sum_{j \in S} u_j p_{ji}(1) = \frac{1}{u_i} (uP)_i = \frac{1}{u_i} u_i = 1.
\]
Therefore, the \( q_{ij}(1) \) define valid 1-step transition probabilities for some Markov chain, which we will denote by \( Y \equiv \{Y_n : n \geq 0\} \). Clearly, \( Y \) has state space \( S \). Assume that \( q_{ij}(n-1) \) gives the \((n-1)\)-step transition probabilities for this Markov chain. Then the \( n \)-step transition probabilities can be written as
\[
P(Y_n = j \mid Y_0 = i) = \sum_{k \in S} P(Y_n = j \mid Y_{n-1} = k, Y_0 = i)P(Y_{n-1} = k \mid Y_0 = i)
\]
\[
= \sum_{k \in S} q_{kj}(1)q_{ik}(n-1)
\]
\[
= \sum_{k \in S} \frac{u_j}{u_k} p_{jk}(1) \frac{u_k}{u_i} p_{ki}(n-1)
\]
\[
= \frac{u_j}{u_i} \sum_{k \in S} p_{jk}(1)p_{ki}(n-1)
\]
\[
= \frac{u_j}{u_i} p_{ji}(n) = q_{ij}(n).
\]
By induction, \( q_{ij}(n) \) gives the \( n \)-step transition probabilities of \( Y \) for all \( n \). For any \( i \neq j \) there is some \( n \) such that \( p_{ji}(n) > 0 \) since \( i \) is accessible from \( j \) in the \( X \) chain. Therefore, \( q_{ij}(n) = (u_j/u_i)p_{ji}(n) > 0 \) so that \( j \) is accessible from \( i \) in the \( Y \) chain. Since \( i \) and \( j \) were arbitrary all states in the \( Y \) chain communicate, so it is irreducible. That state \( i \) is recurrent in the \( Y \) chain follows from
\[
\sum_{n=0}^{\infty} q_{ii}(n) = \sum_{n=0}^{\infty} p_{ii}(n) = \infty,
\]
where the last equality follows because \( i \) is recurrent in the \( X \) chain.
(b) (4 marks) For \( n = 1 \) we have that \( g_{ij}(1) \) is just the probability that starting in state \( i \) the \( Y \) chain transitions to state \( j \), and so

\[
g_{ij}(1) = q_{ij}(1) = \frac{u_j}{u_i} p_{ji}(1).
\]

But \( p_{ji}(1) = P(X_1 = i \mid X_0 = j) = l_{ji}(1) \) so the result follows for \( n = 1 \). Now suppose the result holds for \( n - 1 \). Then for \( n \), by intersecting with all possible values of \( Y_1 \) we have

\[
g_{ij}(n) = P(Y_n = j, Y_{n-1} \neq j, \ldots, Y_1 \neq j \mid Y_0 = i)
\]

\[
= \sum_{k \in S} P(Y_n = j, Y_{n-1} \neq j, \ldots, Y_1 \neq j, Y_1 = k \mid Y_0 = i)
\]

\[
= \sum_{k \neq j} P(Y_n = j, Y_{n-1} \neq j, \ldots, Y_2 \neq j, Y_1 = k \mid Y_0 = i)
\]

\[
= \sum_{k \neq j} P(Y_n = j, Y_{n-1} \neq j, \ldots, Y_2 \neq j, Y_1 = k, Y_0 = i) P(Y_1 = k \mid Y_0 = i)
\]

\[
= \sum_{k \neq j} P(Y_n = j, Y_{n-1} \neq j, \ldots, Y_2 \neq j, Y_1 = k) P(Y_1 = k \mid Y_0 = i)
\]

\[
= \sum_{k \neq j} P(Y_{n-1} = j, Y_{n-2} \neq j, \ldots, Y_1 \neq j, Y_0 = k) P(Y_1 = k \mid Y_0 = i)
\]

\[
= \sum_{k \neq j} g_{kj}(n-1) q_{ik}(1)
\]

\[
= \sum_{k \neq j} \frac{u_j}{u_i} l_{jk}(n-1) \frac{u_k}{u_i} p_{ki}(1)
\]

\[
= \frac{u_j}{u_i} \sum_{k \neq j} P(X_{n-1} = k, X_{n-2} \neq j, \ldots, X_1 \neq j \mid X_0 = j) P(X_1 = i \mid X_0 = k)
\]

\[
= \frac{u_j}{u_i} \sum_{k \neq j} P(X_{n-1} = k, X_{n-2} \neq j, \ldots, X_1 \neq j \mid X_0 = j) P(X_n = i \mid X_{n-1} = k)
\]

\[
= \frac{u_j}{u_i} \sum_{k \neq j} \left[ P(X_{n-1} = k, X_{n-2} \neq j, \ldots, X_1 \neq j \mid X_0 = j) \right]
\]

\[
\times P(X_n = i \mid X_{n-1} = k, X_{n-2} \neq j, \ldots, X_1 \neq j, X_0 = j)
\]

\[
= \frac{u_j}{u_i} \sum_{k \neq j} P(X_n = i, X_{n-1} = k, X_{n-2} \neq j, \ldots, X_1 \neq j \mid X_0 = j)
\]

\[
= \frac{u_j}{u_i} P(X_n = i, X_{n-1} \neq j, X_{n-2} \neq j, \ldots, X_1 \neq j \mid X_0 = j)
\]

\[
= \frac{u_j}{u_i} l_{ji}(n),
\]

where equality 5 follows from the Markov property, equality 6 from time homogeneity, equality 8 from the induction hypothesis, equality 10 from time homogeneity, equality 12 from the induction hypothesis, and equality 13 from time homogeneity.
geneity, and equality 11 from the Markov property. By induction we have our
desired result:
\[ g_{ij}(n) = \frac{u_j}{u_i} l_{ji}(n) \quad \text{for all } n \geq 1. \quad (1) \]

(c) (3 marks) If we sum \( g_{ij}(n) \) over \( n \) we get the probability that the \( Y \) chain ever
visits state \( j \) starting in state \( i \), which is equal to 1 from arguments discussed in
class since the \( Y \) chain is irreducible and recurrent by part(a) (the argument from
class is that the \( Y \) chain will return to state \( i \) infinitely often with probability 1
by recurrence and each time it returns to state \( i \) there is a positive probability of
visiting state \( j \) before its next return to state \( i \) since the chain is irreducible; these
two facts imply that the probability state \( j \) will ever be visited is 1). Summing
\( l_{ji}(n) \) over \( n \) we get the expected number of visits to state \( i \) during a sojourn from
state \( j \) back to state \( j \) in the \( X \) chain, which we denote by \( \rho_i(j) \). Thus, we get
\[ u_i = u_j \rho_i(j) \quad \text{for } i \neq j, \]
and by labelling one of the states in \( S \) as 0 we have
\[ u_i = u_0 \rho_i(0) \quad \text{for all } i \neq 0. \]
This implies that \( \mathbf{u} = \rho_0 \) is unique up to a multiplicative constant (since the
vector \( \rho_0 = (\rho_i(0))_{i \in S} \) is unique).

(d) (2 marks) In the \( X \) chain we have that all states are recurrent. From class notes
we know that either all states are null recurrent or all states are positive recurrent.
If all states are null recurrent then we cannot have \( \sum_{i \in S} u_i < \infty \) because then we
could normalize the vector \( \mathbf{u} \) and it would be a stationary distribution, which we
know from class would imply that all states were positive recurrent. Therefore,
we must have \( \sum_{i \in S} u_i = \infty \). On the other hand if \( \sum_{i \in S} u_i = \infty \) then we could
not have that all states were positive recurrent, because then there would be a
unique stationary distribution and by part(c) that stationary distribution would
have to be a multiple of \( \mathbf{u} \) (because it satisfies the conditions for \( \mathbf{u} \)). But this
is impossible if \( \sum_{i \in S} u_i = \infty \). Therefore, we must have that all states are null
recurrent. So all states are null recurrent if and only if \( \sum_{i \in S} u_i = \infty \).

2. From Sheet. (5 marks) This chain is positive recurrent, and we find the stationary
distribution \( \pi \). From Problem 4 on Assignment 2, the transition matrix for this chain
is given by
\[
P = \begin{bmatrix}
1/2 & 1/2 & 0 & 0 & 0 & \cdots \\
1/3 & 1/3 & 1/3 & 0 & 0 & \cdots \\
1/4 & 1/4 & 1/4 & 1/4 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix},
\]
and the state space is $S = \{0, 1, 2, \ldots\}$. Writing out the equations $\pi = \pi \mathbf{P}$, we have

$$\pi_0 = \sum_{k=0}^{\infty} \frac{\pi_k}{k+2} \quad \text{(2)}$$

$$\pi_n = \sum_{k=n-1}^{\infty} \frac{\pi_k}{k+2} \quad \text{for } n \geq 1. \quad \text{(3)}$$

From (3), subtracting the $(n+1)$th equation from the $n$th equation, we have

$$\pi_n - \pi_{n+1} = \frac{\pi_{n-1}}{n+1} \quad \text{for } n \geq 1$$

which is the same as

$$n\pi_n - (n+1)\pi_{n+1} = \pi_{n-1} - \pi_n \quad \text{for } n \geq 1. \quad \text{(4)}$$

For $m \geq 1$, if we sum both sides of (4) from $n = 1$ to $n = m$, since both sides are telescoping sums, we have

$$\pi_1 - (m+1)\pi_{m+1} = \pi_0 - \pi_m \quad \text{for } m \geq 1. \quad \text{(5)}$$

Now note that the RHS of the equation for $\pi_0$ in (2) and for $\pi_1$ in (3) are the same, so we have that $\pi_0 = \pi_1$. Using this fact in (5) we obtain

$$\pi_{m+1} = \frac{1}{m+1} \pi_m \quad \text{for } m \geq 1.$$

Recursing down we have that

$$\pi_{m+1} = \frac{1}{(m+1)!} \pi_1 = \frac{1}{(m+1)!} \pi_0 \quad \text{for } m \geq 1.$$

Solving for $\pi_0$ using the normalization constraint $\sum_{i=0}^{\infty} \pi_i = 1$ we compute

$$1 = \pi_0 \left( 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \right) = \pi_0 e,$$

and so $\pi_0 = 1/e$. Thus, we have that the stationary distribution is given by

$$\pi_n = \frac{1}{n!e} \quad \text{for } n \geq 0.$$
3. From Sheet. (7 marks)

(a) (3 marks) Conditioning on $X_0$ we have

$$P(X_n = j) = \sum_{i \in S} P(X_n = j \mid X_0 = i) P(X_0 = i) = \sum_{i \in S} p_{ij}(n) P(X_0 = i).$$

Therefore,

$$\lim_{n \to \infty} P(X_n = j) = \lim_{n \to \infty} \sum_{i \in S} p_{ij}(n) P(X_0 = i) = \sum_{i \in S} P(X_0 = i) \lim_{n \to \infty} p_{ij}(n) = \sum_{i \in S} P(X_0 = i) \pi_j = \pi_j,$$

where taking the limit inside the sum in the second equality is justified by an argument from class when we did a similar thing there (when we showed a stationary distribution cannot exist if all states are transient) and the third equality follows from our result in class that $\lim_{n \to \infty} p_{ij}(n) = \pi_j$. The result follows since (also from class) $\pi_j = 1/\mu_j$.

(b) (4 marks) We first show that if a sequence $\{x_n\}_{n=1}^\infty$ has a limit $x$ (i.e., $\lim_{n \to \infty} x_n = x$) then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n x_i = x$$

as well (i.e., the limit of partial averages converges to $x$). Let $\epsilon > 0$ be given. Let $N_0$ be such that $|x_n - x| < \epsilon/2$ for all $n > N_0$, and let

$$N_1 = \frac{2}{\epsilon} \sum_{i=1}^{N_0} |x_i - x|.$$

Then for $n > N \triangleq \max(N_0, N_1)$, we get

$$\left| \frac{1}{n} \sum_{i=1}^n x_i - x \right| = \left| \frac{1}{n} \sum_{i=1}^n (x_i - x) \right| \leq \frac{1}{n} \sum_{i=1}^n |x_i - x|$$

$$= \frac{1}{n} \sum_{i=1}^{N_0} |x_i - x| + \frac{1}{n} \sum_{i=N_0+1}^n |x_i - x|$$

$$< \frac{1}{N_1} \sum_{i=1}^{N_0} |x_i - x| + \frac{1}{n} \sum_{i=N_0+1}^n \frac{\epsilon}{2}$$

$$= \frac{\epsilon}{2} + \frac{n-N_0}{n} \frac{\epsilon}{2}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$
Therefore $\frac{1}{n} \sum_{i=1}^{n} x_i \to x$ as $n \to \infty$. Now letting $x_n = P(X_n = j)$ and $x = \pi_j$ we have that $x_n \to x$ as $n \to \infty$ from part(a) (since $\pi_j = 1/\mu_j$) and so the desired result follows directly.

4. From Sheet. (5 marks) The global balance equations for this Markov chain are $\pi_1 = (1 - \alpha)\pi_1 + \beta\pi_2$ and $\pi_1 + \pi_2 = 1$, with solution

$$\pi = (\pi_1, \pi_2) = \left(\frac{\beta}{\alpha + \beta}, \frac{\alpha}{\alpha + \beta}\right).$$

Now we show that

$$(\alpha + \beta)P^n = (1 - \alpha - \beta)^n \begin{bmatrix} \alpha & -\alpha \\ -\beta & \beta \end{bmatrix} + \begin{bmatrix} \beta & \alpha \\ \alpha & \beta \end{bmatrix};$$

For $n = 1$, the statement reduces to

$$(\alpha + \beta)P = \begin{bmatrix} \alpha + \beta & 0 \\ 0 & \alpha + \beta \end{bmatrix} - (\alpha + \beta) \begin{bmatrix} \alpha & -\alpha \\ -\beta & \beta \end{bmatrix}$$

or

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \alpha & -\alpha \\ -\beta & \beta \end{bmatrix} = \begin{bmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{bmatrix},$$

which is true. Now assume the statement is true for $n - 1$. Then we have

$$(\alpha + \beta)P^n = (\alpha + \beta)P^{n-1}P$$

by the induction hypothesis

$$= (1 - \alpha - \beta)^{n-1} \left[ \begin{bmatrix} \alpha & -\alpha \\ -\beta & \beta \end{bmatrix} + \begin{bmatrix} \beta & \alpha \\ \alpha & \beta \end{bmatrix} \right] \begin{bmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{bmatrix}$$

$$= (1 - \alpha - \beta)^{n-1} \begin{bmatrix} \alpha - \alpha^2 - \alpha\beta & \alpha^2 - \alpha + \alpha\beta \\ -\beta + \alpha\beta + \beta^2 & -\alpha\beta + \beta + \beta^2 \end{bmatrix}$$

$$+ \begin{bmatrix} \beta - \alpha\beta + \alpha\beta & \alpha\beta + \alpha - \alpha\beta \\ \beta - \alpha\beta + \alpha\beta & \alpha\beta + \alpha - \alpha\beta \end{bmatrix}$$

$$= (1 - \alpha - \beta)^{n-1} \begin{bmatrix} \alpha(1 - \alpha - \beta) & -\alpha(1 - \alpha - \beta) \\ -\beta(1 - \alpha - \beta) & \beta(1 - \alpha - \beta) \end{bmatrix} + \begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix}$$

Therefore, the required statement is true by induction. Since $|1 - \alpha - \beta| < 1$, $(1 - \alpha - \beta)^n \to 0$ as $n \to \infty$, and so

$$(\alpha + \beta)P^n \to \begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix}$$
as $n \to \infty$, which is the same as
\[
\begin{align*}
p_{11}(n) &\to \beta/\alpha + \beta = \pi_1, \\
p_{21}(n) &\to \beta/\alpha + \beta = \pi_1, \\
p_{12}(n) &\to \alpha/\alpha + \beta = \pi_2, \\
p_{22}(n) &\to \alpha/\alpha + \beta = \pi_2
\end{align*}
\]
as $n \to \infty$.

5. From Sheet. (9 marks) If $X_n > 0$, the number of customers in the system in the next time unit is
\[
X_{n+1} = X_n - 1 + Z_{n+1},
\]
while if $X_n = 0$, we have
\[
X_{n+1} = Z_{n+1}.
\]
Thus the transition probabilities are
\[
P(X_{n+1} = j \mid X_n = 0) = f(j) \quad \text{for } j \geq 0,
\]
and for $i > 0$,\[
P(X_{n+1} = j \mid X_n = i) = P(Z_{n+1} = j - i + 1) = f(j - i + 1) \quad \text{for } j \geq i - 1.
\]
We can write down the transition probability matrix as
\[
P = \begin{bmatrix}
0 & f(0) & f(1) & f(2) & \cdots \\
1 & f(0) & f(1) & f(2) & \cdots \\
2 & f(0) & f(1) & \cdots \\
3 & f(0) & \cdots \\
\vdots & \vdots & \vdots \\
\end{bmatrix}
\]
To derive $G(s)$, the probability generating function of the stationary distribution $\pi = (\pi_0, \pi_1, \ldots)$, we write down the global balance equations. These are
\[
\begin{align*}
\pi_0 &= \pi_0 f(0) + \pi_1 f(0) \\
\pi_1 &= \pi_0 f(1) + \pi_1 f(1) + \pi_2 f(0) \\
\pi_2 &= \pi_0 f(2) + \pi_1 f(2) + \pi_2 f(1) + \pi_3 f(0) \\
&\vdots
\end{align*}
\]
In general,
\[ \pi_j = \pi_0 f(j) + \sum_{i=1}^{j+1} \pi_i f(j + 1 - i) \quad \text{for } j \geq 0. \]

Multiplying by \( s^j \) and summing over \( j \) gives
\[ \sum_{j=0}^\infty s^j \pi_j = \pi_0 \sum_{j=0}^\infty s^j f(j) + \sum_{j=0}^\infty s^j \sum_{i=1}^{j+1} \pi_i f(j + 1 - i), \]
or
\[
G(s) = \pi_0 G_Z(s) + \frac{1}{s} \sum_{j=0}^\infty s^{j+1} \sum_{i=1}^{j+1} \pi_i f(j + 1 - i)
\]
\[
= \pi_0 G_Z(s) + \frac{1}{s} \sum_{j=1}^\infty s^j \sum_{i=1}^{j} \pi_i f(j - i)
\]
\[
= \pi_0 G_Z(s) + \frac{1}{s} \left[ \sum_{j=0}^\infty s^j \sum_{i=0}^{j} \pi_i f(j - i) - \pi_0 f(0) - \sum_{j=1}^\infty \pi_0 s^j f(j) \right]
\]
\[
= \pi_0 G_Z(s) + \frac{1}{s} \left[ G(s) G_Z(s) - \pi_0 G_Z(s) \right].
\]

Solving for \( G(s) \) we get
\[
G(s) \left( 1 - \frac{1}{s} G_Z(s) \right) = \pi_0 G_Z(s) \left( 1 - \frac{1}{s} \right)
\]
or
\[
G(s) = \frac{\pi_0 G_Z(s) (s - 1)}{s - G_Z(s)}.
\]

Finally, using the fact that (since the stationary distribution exists), \( \lim_{s \uparrow 1} G(s) = 1 \), we obtain
\[
1 = \lim_{s \uparrow 1} \frac{\pi_0 G_Z(s) (s - 1)}{s - G_Z(s)}
\]
\[
= \lim_{s \uparrow 1} \frac{\pi_0 (G'_Z(s) (s - 1) + G_Z(s))}{1 - G'_Z(s)} \quad \text{using l'Hospital's rule}
\]
\[
= \frac{\pi_0 (\mu(0) + 1)}{1 - \mu} \Rightarrow \pi_0 = 1 - \mu.
\]

Therefore, the result follows.

\*6. From Sheet. (8 marks) We let \( Y = \{Y_n : n \geq 0\} \) be an independent copy of \( X \) with initial distribution \( \pi \). Following the class notes, we have
\[
|p_{ij}(n) - \pi_j| \leq P(T > n),
\]
where $T$ is the first time the two chains $X$ and $Y$ couple (it’s ok to start at this point in your solution). Now, using the law of total probability, write

\[
P(T > n) = P(T > n \mid T > n - 1)P(T > n - 1) + P(T > n \mid T \leq n - 1)P(T \leq n - 1)
\]

\[
= P(T > n \mid T > n - 1)P(T > n - 1),
\]

since $P(T > n \mid T \leq n - 1) = 0$. Recursively, then, we have

\[
P(T > n) = \prod_{r=0}^{n-1} P(T > r + 1 \mid T > r),
\]

since $P(T > 0) = 1$. Now note that no matter where the two chains are in the state space $S$ at any given time, there is a positive probability that they are coupled in the next step since $p_{ij} > 0$ for all $i, j \in S$. It is easy to see that the probability that they are coupled in the next step is lower bounded by $\epsilon^2$, where $\epsilon = \min_{i,j \in S} p_{ij}$, where $\epsilon$ is positive because $p_{ij} > 0$ for all $i, j \in S$ and the state space $S$ is finite. On the other hand, $P(T > r + 1 \mid T > r)$ is the probability that they do not couple at time $r + 1$ given that at time $r$ they have not yet coupled. Therefore,

\[
P(T > r + 1 \mid T > r) \leq 1 - \epsilon^2 < 1.
\]

Thus, choosing $\lambda \in (1 - \epsilon^2, 1)$, we have

\[
|p_{ij}(n) - \pi_j| < \lambda^n.
\]