Stochastic Processes

Assignment #4, Solutions

Total Marks: 33 for both 455 and 855.

1. From Sheet. (10 marks)

(a) (3 marks) The given expression for \( G^n \) is clearly correct for \( n = 1 \). Assume it is true for \( n \). Direct matrix multiplication then yields

\[
G^{n+1} = G^n G
\]

\[
= \begin{bmatrix}
(\lambda + \mu)^n & (-1)^n (\lambda + \mu)^{n-1} \\
(\lambda + \mu)^n & (-1)^n (\lambda + \mu)^{n-1}
\end{bmatrix}
\begin{bmatrix}
-\mu & \mu \\
\lambda & -\lambda
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\mu (\lambda + \mu)^{n-1}(-1)^n (\mu + \lambda) & \mu (\lambda + \mu)^{n-1}(-1)^n (\mu + \lambda) \\
\lambda (\lambda + \mu)^{n-1}(-1)^n (\mu + \lambda) & \lambda (\lambda + \mu)^{n-1}(-1)^n (\mu + \lambda)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(-1)^{n+1} (\lambda + \mu)^n & (-1)^n (\lambda + \mu)^n \\
(-1)^n (\lambda + \mu)^n & (-1)^{n+1} (\lambda + \mu)^n
\end{bmatrix},
\]

and so the given form for \( G^n \) is true for all \( n \geq 1 \).

(b) (7 marks) Noting that \((tG)^0/0! = G^0 = I\), the 2 \times 2 identity matrix, we have

\[
p_{11}(t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n!} \frac{1}{\lambda+\mu} = 1 + \frac{\mu}{\lambda+\mu} \sum_{n=1}^{\infty} \frac{(-t(\lambda+\mu))^n}{n!}
\]

\[
= 1 + \frac{\mu}{\lambda+\mu} \left( e^{-t(\lambda+\mu)} - 1 \right) = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-t(\lambda+\mu)}.
\]

Similarly,

\[
p_{22}(t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^n \lambda (\lambda + \mu)^{n-1}}{n!} = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-t(\lambda+\mu)}
\]

\[
p_{12}(t) = \frac{-\mu}{\lambda+\mu} \sum_{n=1}^{\infty} \frac{(-t(\lambda+\mu))^n}{n!} = \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-t(\lambda+\mu)}
\]

\[
p_{21}(t) = \frac{-\lambda}{\lambda+\mu} \sum_{n=1}^{\infty} \frac{(-t(\lambda+\mu))^n}{n!} = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-t(\lambda+\mu)}.
\]
Thus,
\[ P(t) = \frac{1}{\lambda + \mu} \begin{bmatrix} \lambda + \mu e^{-t(\lambda + \mu)} & \mu - \mu e^{-t(\lambda + \mu)} \\ \lambda - \lambda e^{-t(\lambda + \mu)} & \mu + \lambda e^{-t(\lambda + \mu)} \end{bmatrix}. \]

Clearly, \( P(0) = I \), and direct differentiation yields
\[ P'(t) = \frac{1}{\lambda + \mu} \begin{bmatrix} -\mu(\lambda + \mu)e^{-t(\lambda + \mu)} & \mu(\lambda + \mu)e^{-t(\lambda + \mu)} \\ \lambda(\lambda + \mu)e^{-t(\lambda + \mu)} & -\lambda(\lambda + \mu)e^{-t(\lambda + \mu)} \end{bmatrix}. \]

On the other hand, it is straightforward to check that
\[ GP(t) = \frac{1}{\lambda + \mu} \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix} \begin{bmatrix} \lambda + \mu e^{-t(\lambda + \mu)} & \mu - \mu e^{-t(\lambda + \mu)} \\ \lambda - \lambda e^{-t(\lambda + \mu)} & \mu + \lambda e^{-t(\lambda + \mu)} \end{bmatrix} = P'(t). \]

Thus, \( P(t) \) satisfies the backward equations \( P'(t) = GP(t) \) with initial condition \( P(0) = I \). Finally, it is easy to see that
\[ \lim_{t \to \infty} P(t) = \frac{1}{\lambda + \mu} \begin{bmatrix} \lambda & \mu \\ \lambda & \mu \end{bmatrix}. \]

On the other hand, the stationary distribution \( \pi = (\pi_1, \pi_2) \) satisfies the global balance equations \( \pi G = 0 \) and \( \pi_1 + \pi_2 = 1 \), or
\[ \mu \pi_1 = \lambda \pi_2 \quad \text{and} \quad \pi_1 + \pi_2 = 1. \]

Solving the above gives \( \pi_1(1 + \mu/\lambda) = 1 \), or \( \pi_1 = \lambda/(\lambda + \mu) \), which then gives \( \pi_2 = \mu/(\lambda + \mu) \). Thus, we see that
\[ \lim_{t \to \infty} P(t) = \begin{bmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{bmatrix}, \]

as expected.
2. From Sheet. (10 marks)

(a) (6 marks) Following the hint, for \( i \neq k \), we have

\[
\gamma_i(k) = \mathbb{E} \left[ \sum_{n=0}^{\infty} U_n I_{\{Z_n = i, T > n\}} \bigg| X(0) = k \right]
\]

\[
= \sum_{n=0}^{\infty} \mathbb{E} [U_n I_{\{Z_n = i, T > n\}} \bigg| X(0) = k] = \sum_{n=0}^{\infty} \mathbb{E} [U_n I_{\{Z_n = i, T > n\}} \bigg| I_{\{Z_n = i, T > n\}} = 1, X(0) = k] \\
\times P(I_{\{Z_n = i, T > n\}} = 1 \bigg| X(0) = k)
\]

\[
= \sum_{n=0}^{\infty} \mathbb{E} [U_n \bigg| Z_n = i, T > n, X(0) = k] P(Z_n = i, T > n \bigg| X(0) = k).
\]

Now \( \mathbb{E} [U_n \bigg| Z_n = i, T > n, X(0) = k] \) can be written as

\[
\mathbb{E} [U_n \bigg| Z_n = i, T > n, X(0) = k] = \mathbb{E} [U_n \bigg| Z_n = i, Z_{n-1} \neq k, \ldots, Z_1 \neq k, Z_0 = k]
\]

\[
= \mathbb{E} [U_n \bigg| Z_n = i] = \frac{1}{v_i}
\]

and so we have

\[
\gamma_i(k) = \frac{1}{v_i} \sum_{n=0}^{\infty} P(Z_n = i, T > n \bigg| X(0) = k) = \frac{1}{v_i} \mathbb{E} \left[ \sum_{n=0}^{\infty} I_{\{Z_n = i, T > n\}} \bigg| X(0) = k \right].
\]

But \( \sum_{n=0}^{\infty} I_{\{Z_n = i, T > n\}} \) given \( X(0) = k \) is exactly equal to the number of times state \( i \) is visited in the \( Z \) chain during a sojourn from state \( k \) back to state \( k \). Hence,

\[
\gamma_i(k) = \frac{\rho_i(k)}{v_i},
\]

as desired.

(b) (4 marks) From part(a), we have that \( \rho_i(k) = v_i \gamma_i(k) \). Also, we know from class that the row vector \( \rho(k) \) satisfies \( \rho(k) = \rho(k)P \), where \( P \) is the one-step transition matrix of the embedded chain \( Z \), with entries \( p_{ij} = q_{ij}/v_i \) for \( i \neq j \) and \( p_{ii} = 0 \). Therefore, for each \( j \in S \), we have

\[
\rho_j(k) = \sum_{i \neq j} \rho_i(k) p_{ij} \quad \Rightarrow \quad v_j \gamma_j(k) = \sum_{i \neq j} v_i \gamma_i(k) \frac{q_{ij}}{v_i}
\]

\[
\Rightarrow \sum_{i \neq j} \gamma_i(k) q_{ij} - \gamma_j(k) v_j = 0
\]

\[
\Rightarrow \sum_{i \in S} \gamma_i(k) q_{ij} = 0.
\]
Therefore, $\gamma(k)G = 0$. Since $X$ has a unique stationary distribution $\pi$ it must be proportional to $\gamma(k)$; that is,

$$
\pi_i = \frac{\gamma_i(k)}{\sum_{j \in S} \gamma_j(k)}.
$$

In particular, $\sum_{j \in S} \gamma_j(k)$ must be finite, and it is equal to the mean return time $\mu_k$. So we have $\pi_i = \gamma_i(k)/\mu_k$ for all $i \in S$. Setting $i = k$, and recalling that $\gamma_k(k) = 1/v_k$, we obtain

$$
\pi_k = \frac{1}{v_k \mu_k},
$$

as desired.

3. From Sheet. (7 marks) If $i$ is a transient state let $f_i$ be the probability that we ever return to state $i$ given that we are currently in state $i$. By time-homogeneity this probability is the same every time we enter state $i$, and by transience $f_i < 1$. Thus, if we start in state $i$, the total number of times we are in state $i$ follows a Geometric distribution with parameter $1 - f_i$. That is, if $N_i$ denotes the total number of exponential holding times in state $i$, then

$$
P(N_i = k \mid X(0) = i) = f_i^{k-1}(1 - f_i) \quad \text{for } k = 1, 2, \ldots
$$

Let $v_i$ be the parameter of the exponential holding time in state $i$. Then given $N_i = k$, the distribution of the total time spent in state $i$ has a Gamma($k, v_i$) distribution (the sum of $k$ independent Exponential($v_i$) random variables). If $T_i$ denotes the total time spent in state $i$, we may compute the cdf of $T_i$ given $X(0) = i$ by conditioning on $N_i$:

$$
P(T_i \leq t \mid X(0) = i) = \sum_{k=1}^{\infty} P(T_i \leq t \mid X(0) = i, N_i = k)P(N_i = k \mid X(0) = i)
$$

$$
= \sum_{k=1}^{\infty} \int_0^t \frac{v_i^k}{(k-1)!} x^{k-1} e^{-v_i x} dx f_i^{k-1}(1 - f_i)
$$

$$
= (1 - f_i) v_i \int_0^t \left[ \sum_{k=1}^{\infty} \frac{(v_i f_i x)^{k-1}}{(k-1)!} \right] e^{-v_i x} dx
$$

$$
= (1 - f_i) v_i \int_0^t e^{v_i f_i x} e^{-v_i x} dx
$$

$$
= (1 - f_i) v_i \int_0^t e^{-(1 - f_i)v_i x} dx
$$

$$
= (1 - f_i) v_i \left. \frac{e^{-(1 - f_i)v_i x}}{(1 - f_i)v_i} \right|_0^t = 1 - e^{-(1 - f_i)v_i t}.
$$
But this is the cdf of an exponential distribution with parameter \((1 - f_i)v_i\), so the result is proven.

4. From Sheet. (6 marks)

(a) (3 marks) The state space is

\[ S = \left\{ n = (n_1, \ldots, n_r) : \sum_{i=1}^{r} n_i = N \text{ and each } n_i \text{ is a nonnegative integer} \right\} . \]

Letting \( e_i \) denote the \( r \)-dimensional vector of all zeroes except with a one in position \( i \), for \( i = 1, \ldots, r \), then from state \( n \) the possible transitions are to states of the form \( n - e_i + e_j \) for \( i \neq j \), indicating that a customer has left server \( i \) and gone to server \( j \). Note that if the transition from \( n \) to \( n - e_i + e_j \) is possible then so is the transition from \( n - e_i + e_j \) to \( n \). The infinitesimal rates of the chain are

\[ q_{n,n-e_i+e_j} = \frac{\mu}{r-1} , \]

for all \( n \in S \) such that \( n - e_i + e_j \) is also in \( S \).

(b) (3 marks) We find the stationary distribution as the solution to the local balance equations. The local balance equations are given by

\[ \pi_n q_{n,n-e_i+e_j} = \pi_{n-e_i+e_j} q_{n-e_i+e_j,n} \]

for all \( n \) such that both \( n \) and \( n - e_i + e_j \) are in \( S \). From the infinitesimal rates given in part (a), these reduce to \( \pi_n = \pi_{n-e_i+e_j} \). That is, setting \( \pi_n = \frac{1}{|S|} \) for all \( n \in S \) will satisfy the local balance equations (the discrete uniform distribution over \( S \)). This gives the stationary probabilities.