Stochastic Processes

Assignment #4, Solutions

Total Marks: 29 for both 455 and 855.

1. From Sheet. (5 marks) First suppose that $X$ is time-reversible. Then local balance is satisfied: $\pi_i p_{ij} = \pi_j p_{ji}$ for all $i, j \in S$. Then

$$\langle x, Py \rangle = \sum_{i \in S} x_i (Py)_i \pi_i$$

$$= \sum_{i \in S} x_i \left( \sum_{j \in S} p_{ij} y_j \right) \pi_i$$

$$= \sum_{i \in S} \sum_{j \in S} x_i y_j \pi_j p_{ji} \quad \text{(by local balance)}$$

$$= \sum_{j \in S} \left( \sum_{i \in S} p_{ji} x_i \right) y_j \pi_j$$

$$= \sum_{j \in S} (Px)_j y_j \pi_j$$

$$= \langle Px, y \rangle.$$ 

Now suppose that $\langle x, Py \rangle = \langle Px, y \rangle$ for all $x, y \in L^2(\pi)$, and fix $i, j \in S$ with $i \neq j$. Set $x = e_j$ and $y = e_i$, where $e_k$ is the $|S|$-vector of all zeroes except for a one in the $k$th component. Then $Px$ is the $j$th column of $P$ and $Py$ is the $i$th column of $P$, and so $\langle Px, y \rangle = \pi_i p_{ij}$ and $\langle x, Py \rangle = \pi_j p_{ji}$. Thus, local balance is satisfied since $i$ and $j$ were arbitrary, and so $X$ is time-reversible.

2. From Sheet. (6 marks)

(a) (2 marks) We check to see if the local balance equations $\pi_i p_{ij} = \pi_j p_{ji}$ are satisfied for all $i, j \in S$ with the given $a_{ij}$. We have

$$\pi_i p_{ij} = \pi_i q_{ij} a_{ij} = \frac{\pi_i q_{ij} \pi_j q_{ji}}{\pi_i q_{ij} + \pi_j q_{ji}} = \pi_j q_{ji} a_{ji} = \pi_j p_{ji}.$$ 

So the local balance equations are satisfied, which implies that $\pi$ is the stationary distribution of the Markov chain with the transition matrix $P$. 


(b) (4 marks) For a general symmetric matrix $B$ satisfying $a_{ij} \leq 1$ for all $i, j$ we have that

\[
\pi_i q_{ij} \min \left( \frac{\pi_j q_{ji}}{\pi_i q_{ij}}, 1 \right) \times b_{ij} = \pi_j q_{ji} \min \left( \frac{\pi_i q_{ij}}{\pi_j q_{ji}}, 1 \right) \times b_{ji}
\]

\[
\iff \ 
\pi_i q_{ij} = \pi_j q_{ji} \min \left( \frac{\pi_i q_{ij}}{\pi_j q_{ji}}, 1 \right) \times b_{ij} \iff b_{ij} = b_{ji}
\]

which is true since $B$ is symmetric, and the second equivalence follows because

\[
\pi_i q_{ij} \min \left( \frac{\pi_j q_{ji}}{\pi_i q_{ij}}, 1 \right) = \pi_j q_{ji} \min \left( \frac{\pi_i q_{ij}}{\pi_j q_{ji}}, 1 \right)
\]

(this is the Metropolis-Hastings algorithm discussed in class). To see that the $a_{ij}$ from part(a) are a special case of the $a_{ij}$ from part(b), set

\[
b_{ij} = \max \left( \frac{\pi_i q_{ij}}{\pi_i q_{ij} + \pi_j q_{ji}}, \frac{\pi_j q_{ji}}{\pi_i q_{ij} + \pi_j q_{ji}} \right).
\]

Then if $\pi_j q_{ji} \leq \pi_i q_{ij}$ we get that $a_{ij}$ from part(b) is given by

\[
a_{ij} = \frac{\pi_j q_{ji}}{\pi_i q_{ij}} \times \frac{\pi_i q_{ij}}{\pi_i q_{ij} + \pi_j q_{ji}} = \frac{\pi_j q_{ji}}{\pi_i q_{ij} + \pi_j q_{ji}},
\]

which is $a_{ij}$ from part(a). If $\pi_j q_{ji} > \pi_i q_{ij}$ we get that $a_{ij}$ from part(b) is given by

\[
a_{ij} = 1 \times \frac{\pi_j q_{ji}}{\pi_i q_{ij} + \pi_j q_{ji}} = \frac{\pi_j q_{ji}}{\pi_i q_{ij} + \pi_j q_{ji}},
\]

which is again the $a_{ij}$ from part(a). Note that the matrix $B$ as defined by the $b_{ij}$ in (1) is a symmetric matrix.

3. From Sheet. (7 marks)

(a) (3 marks) We check local balance. The local balance equations are

\[
\pi_i \sum_{k \in V} w_{ik} = \pi_j \sum_{k \in V} w_{jk}
\]

for all $i, j \in V$, and both sides are 0 if $(i, j)$ is not an edge in the graph, in which case both sides have positive transition probabilities. Since $w_{ij} = w_{ji}$ for all $i, j$, it is easy to see that setting $\pi_i = c \sum_{k \in V} w_{ik}$ for all $i \in V$ satisfies the above equations, where $c$ is a normalizing constant. From the normalization constraint $\sum_{i \in V} \pi_i = 1$ we obtain

\[
\pi_i = \frac{\sum_{k \in V} w_{ik}}{\sum_{v \in V} \sum_{k \in V} w_{vk}}.
\]
(b) (4 marks) Take the squares on the chess board to be the vertices of a graph, with an edge between a pair of squares if the knight can move from one of the squares in the pair to the other square in the pair in one move. Give weight one to every edge. Then the knight moving randomly about the chess board is the same as a particle moving among the vertices of the corresponding graph with the given weights. Also, note that this graph is connected because no matter where the knight is, it can move to any square in a finite number of moves (to see this it is sufficient to see that the knight can move to any square adjacent to its current position in three moves). If we label one of the corner squares as vertex 0, then the mean return time to vertex 0 is \(1/\pi_0\), where \(\pi\) is the stationary distribution from part(a). From part(a), to compute \(\pi\), we need, for each square, the total weight from the edges connected to that square. For each square, this total weight is number of moves the knight can legally make from that square. We distinguish between squares on the perimeter of the grid (border squares), squares that are on the perimeter of the inner 6 \(\times\) 6 grid of non-border squares (inner-border squares), and the remaining 16 squares of the innermost 4 \(\times\) 4 grid (inner squares). From any inner square the knight can make all of its 8 possible legal moves. There are 20 inner-border squares. From 16 of them the knight can make 6 moves and from 4 of them the knight can make 4 moves. There are 28 border squares. From 16 of them the knight can make 4 moves, from 8 of them the knight can make 3 moves, and from 4 of them (the corner squares) the knight can make 2 moves. Thus, the total weight in the denominator of \(\pi_i\) from part(a) is

\[
\]

The total weight from the edges connected to a corner square is 2. Therefore, \(\pi_0 = 2/336 = 1/168\), and the mean time for the knight to return to the corner square it started in is 168 moves.

4. From Sheet. (7 marks) If \(i\) is a transient state let \(f_i\) be the probability that we ever return to state \(i\) given that we are currently in state \(i\). By time-homogeneity this probability is the same every time we enter state \(i\), and by transience \(f_i < 1\). Thus, if we start in state \(i\), the total number of times we are in state \(i\) follows a Geometric distribution with parameter \(1 - f_i\). That is, if \(N_i\) denotes the total number of exponential holding times in state \(i\), then

\[
P(N_i = k \mid X(0) = i) = f_i^{k-1}(1 - f_i) \quad \text{for } k = 1, 2, \ldots
\]
Let \( v_i \) be the parameter of the exponential holding time in state \( i \). Then given \( N_i = k \), the distribution of the total time spent in state \( i \) has a Gamma\((k, v_i)\) distribution (the sum of \( k \) independent Exponential\((v_i)\) random variables). If \( T_i \) denotes the total time spent in state \( i \), we may compute the cdf of \( T_i \) given \( X(0) = i \) by conditioning on \( N_i \):

\[
P(T_i \leq t \mid X(0) = i) = \sum_{k=1}^{\infty} P(T_i \leq t \mid X(0) = i, N_i = k) P(N_i = k \mid X(0) = i)
\]

\[
= \sum_{k=1}^{\infty} \int_0^t \frac{v_i^k}{(k-1)!} x^{k-1} e^{-v_i x} dx f_i^{k-1}(1 - f_i)
\]

\[
= (1 - f_i) v_i \int_0^t \left[ \sum_{k=1}^{\infty} \frac{(v_i f_i x)^{k-1}}{(k-1)!} \right] e^{-v_i x} dx
\]

\[
= (1 - f_i) v_i \int_0^t e^{v_i f_i x} e^{-v_i x} dx
\]

\[
= (1 - f_i) v_i \int_0^t e^{(1 - f_i) v_i x} dx
\]

\[
= (1 - f_i) v_i \left. \frac{-e^{(1 - f_i) v_i x}}{(1 - f_i) v_i} \right|_0^t = 1 - e^{-(1 - f_i) v_i t}.
\]

But this is the cdf of an exponential distribution with parameter \((1 - f_i) v_i\), so the result is proven.

5. From Sheet. (4 marks) The state space is \( \{1, 2, \ldots, N\} \) since each time a new individual gets infected the process jumps from its current state to its current state plus one, unless it is in state \( N \). If all individuals are infected it will stay in state \( N \) forever. Suppose the process is currently in state \( i \), for \( i < N \). So there are \( i \) infected individuals and \( N - i \) noninfected individuals. So there are \( i(N - i) \) different pairs of individuals in which one is infected and the other is not. Then each time a contact occurs the probability that a new individual will get infected is

\[
p_i = \frac{i(N - i)p}{\binom{N}{2}},
\]

independently from contact to contact. So the number of contacts until a new individual gets infected is Geometrically distributed with parameter \( p_i \). Following the solution of problem 4, the total time spent in state \( i \) (which is the time from entering state \( i \) until the time a new individual gets infected) is Exponentially distributed with rate \( \lambda p_i \). So the transition rate from state \( i \) to state \( i + 1 \) is \( q_{i,i+1} = \lambda p_i \). All other rates \( q_{ij} \) are equal to 0, for \( i \neq j \) (including the case when \( i = N \)).