Stochastic Processes

Assignment #5, Solutions

Total Marks: 33 for both 455 and 855.

1. From Sheet. (7 marks)

(a) (3 marks) The problem is to compute the probability that there are \( r \) arrivals in the \( N_2 \) process during an Exponential(\( \lambda_1 \)) distributed length of time. Let \( T \) denote a random variable with an Exponential(\( \lambda_1 \)) distribution. Given that \( T = t \) the probability that there are \( r \) arrivals in the \( N_2 \) process during the length of time \( T \) is the Poisson probability

\[
P(r \text{ arrivals in } N_2 \text{ process during time } T \mid T = t) = \frac{(\lambda_2 t)^r}{r!} e^{-\lambda_2 t}.
\]

Then by the law of total probability we have

\[
P(r \text{ arrivals in } N_2 \text{ process during time } T) = \int_0^\infty \frac{(\lambda_2 t)^r}{r!} e^{-\lambda_2 t} \lambda_1 e^{-\lambda_1 t} dt
\]

\[
= \frac{\lambda_1 \lambda_2^r}{(\lambda_1 + \lambda_2)^{r+1}} \int_0^\infty \frac{(\lambda_1 + \lambda_2)^{r+1}}{r!} t^r e^{-(\lambda_1 + \lambda_2)t} dt
\]

\[
= \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right) \left( \frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^r,
\]

where the integral above is equal to 1 because we can recognize the integrand as a Gamma\((r + 1, \lambda_1 + \lambda_2)\) density.

(b) (4 marks) Conditional on \( N_1(1) = 1 \) the distribution of the arrival time of the first arrival in the \( N_1 \) process is Uniform(0,1), so if \( T_2 \) and \( T \) denote the arrival time of the first arrival in the \( N_2 \) and \( N \) process, respectively, we have conditional on \( N_1(1) = 1 \), that \( T = \min(U, T_2) \), where \( U \sim \text{Uniform}(0,1) \) and, since the \( N_2 \) process is independent of the \( N_1 \) process, \( T_2 \sim \text{Exponential}(\lambda_2) \) and is independent of \( U \). We first obtain the distribution function of \( \min(U, T_2) \). We have firstly that \( \min(U, T_2) \in (0, 1) \). For \( x \in (0, 1) \), we first compute

\[
P(\min(U, T_2) > x) = P(U > x, T_2 > x) = P(U > x)P(T_2 > x) = (1 - x)e^{-\lambda_2 x}
\]
giving the distribution function to be $1-(1-x)e^{-\lambda x}$, and so the pdf of $\min(U, T_2)$ is given by $e^{-\lambda x}(1 \pm (1-x)\lambda x)I_{(0,1)}(x)$. Then

$$E[T \mid N_1(1) = 1] = E[\min(U, T_2)] = \int_0^1 x(1 + \lambda x(1-x)e^{-\lambda x}dx.$$  

To compute the integral above we do the preliminary integration by parts computations

$$\int_0^1 xe^{-\lambda x}dx = \left. -\frac{x e^{-\lambda x}}{\lambda} \right|_0^1 + \frac{1}{\lambda} \int_0^1 e^{-\lambda x}dx = -\frac{e^{-\lambda x}}{\lambda} \bigg|_0^1 = \frac{1 - e^{-\lambda x}(1 + \lambda x)}{\lambda^2}$$

and

$$\int_0^1 x^2e^{-\lambda x}dx = \left. -\frac{x^2 e^{-\lambda x}}{\lambda} \right|_0^1 + \frac{2}{\lambda} \int_0^1 e^{-\lambda x}dx = -\frac{e^{-\lambda x}}{\lambda^2} + \frac{2}{\lambda} \left(1 - \frac{e^{-\lambda x}(1 + \lambda x)}{\lambda^2} \right).$$

Then we have

$$E[T \mid N_1(1) = 1] = (1 + \lambda)(1 - \frac{e^{-\lambda x}}{\lambda} - \frac{e^{-\lambda x}}{\lambda} + \frac{2}{\lambda} \left(1 - \frac{e^{-\lambda x}(1 + \lambda x)}{\lambda^2} \right))$$

$$= e^{-\lambda x} + (\lambda - 1) \frac{1 - e^{-\lambda x}(1 + \lambda x)}{\lambda^2}$$

$$= e^{-\lambda x} + \frac{1}{\lambda} - \frac{e^{-\lambda x}}{\lambda^2} - \frac{1 - e^{-\lambda x}}{\lambda^2} + \frac{e^{-\lambda x}}{\lambda^2} + \frac{e^{-\lambda x}}{\lambda^2}$$

$$= \frac{1}{\lambda^2} - \frac{1}{\lambda^2} + \frac{e^{-\lambda x}}{\lambda^2} = \frac{\lambda - 1 + e^{-\lambda x}}{\lambda^2}.$$

2. From Sheet. (8 marks)

(a) (2 marks) Let type 1 events be those for which the CF is greater than 2. Then the probability that a given event is type 1 is $p_1 = e^{-2c}$, independent of the time of the event. The time until the first type 1 event is exponential with parameter $\lambda p_1$ and mean $1/(\lambda p_1) = 1/(\lambda e^{-2c})$. 

(b) (3 marks) First, condition on \(N(t)\). Given \(N(t) = n\), we have

\[
E[CE(t) \mid N(t) = n] = E\left[\sum_{i=1}^{N(t)} CF_i e^{-r(t-S_i)} \mid N(t) = n\right]
\]

\[
= E\left[\sum_{i=1}^{n} CF_i e^{-r(t-S_i)} \mid N(t) = n\right]
\]

\[
= \sum_{i=1}^{n} E[CF_i e^{-r(t-U_i)}] \quad \text{where } U_i \sim \text{Uniform}(0, t)
\]

\[
= \sum_{i=1}^{n} E[CF_i] E[e^{-r(t-U_i)}]
\]

\[
= \frac{n}{ct} \int_0^t e^{-r(t-s)} ds
\]

\[
= \frac{n e^{-r(t-s)}}{ct - r} \bigg|_0^t
\]

\[
= \frac{n(1 - e^{-rt})}{rct}.
\]

Unconditioning, we have

\[
E[CE(t)] = \frac{E[N(t)](1 - e^{-rt})}{rct} = \frac{\lambda t(1 - e^{-rt})}{rct} = \frac{\lambda(1 - e^{-rt})}{rc}.
\]

(c) (3 marks) Let \(A\) denote the event that a Second Age of Cuteness comes. Conditioned on \(T = t\) and \(N(t) = n\), event \(A\) occurs if and only if \(\max(X_1, \ldots, X_n) > a\), where \(X_1, \ldots, X_n\) are independent and identically distributed exponential random variables with parameter \(c\). We have

\[
P(A \mid T = t, N(t) = n) = P(\max(X_1, \ldots, X_n) > a)
\]

\[
= 1 - P(\max(X_1, \ldots, X_n) \leq a)
\]

\[
= 1 - P(X_1 \leq a, \ldots, X_n \leq a)
\]

\[
= 1 - P(X_1 \leq a)^n
\]

\[
= 1 - (1 - e^{-ca})^n.
\]
Unconditioning, we have

\[
P(A) = \int_0^\infty \sum_{n=0}^\infty \left(1 - (1 - e^{-ca})^n\right) \frac{(\lambda t)^n}{n!} e^{-\lambda t} \mu e^{-\mu t} dt
\]

\[
= \int_0^\infty \mu e^{-\lambda t} \left(e^\lambda - e^{\lambda(1-e^{-ca})}\right) dt
\]

\[
= \int_0^\infty \left(\mu e^{-\mu t} - \mu e^{-(\mu + \lambda e^{-ca})t}\right) dt
\]

\[
= 1 - \frac{\mu}{\mu + \lambda e^{-ca}}
\]

\[
= \frac{\lambda e^{-ca}}{\mu + \lambda e^{-ca}}.
\]

3. From Sheet. (4 marks) We define a launch (event) to be of type 1 if it occurred before time \(s\) and the satellite was still in the air at time \(t\). Letting \(N_1(t)\) be the number of type 1 events up to time \(t\), we wish to find \(P(N_1(t) = 0)\). By Proposition 5.3, \(N_1(t)\) has a Poisson distribution with mean \(\lambda \int_0^t p_1(u) du\), where

\[
p_1(u) = P\text{(event occurring at time } u \text{ is of type 1)}
\]

\[
= \begin{cases} 
1 - G(t - u) & \text{if } u < s \\
0 & \text{if } u \geq s
\end{cases}
\]

Therefore,

\[
\lambda \int_0^t p_1(u) du = \lambda \int_0^s (1 - G(t - u)) du = \lambda s - \lambda \int_0^s G(t - u) du
\]

and

\[
P(N_1(t) = 0) = \exp \left\{ - \left(\lambda s - \lambda \int_0^s G(t - u) du\right) \right\}.
\]
4. From Sheet. (8 marks)

(a) (3 marks) The given expression for \( G^n \) is clearly correct for \( n = 1 \). Assume it is true for \( n \). Direct matrix multiplication then yields

\[
G^{n+1} = G^n G = \begin{bmatrix} (-1)^n \mu (\lambda + \mu)^{n-1} & (-1)^{n-1} \mu (\lambda + \mu)^{n-1} \\ (-1)^{n-1} \lambda (\lambda + \mu)^{n-1} & (-1)^n \lambda (\lambda + \mu)^{n-1} \end{bmatrix} \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}
\]

\[
= \begin{bmatrix} \mu (\lambda + \mu)^{n-1} (-1)^n (\mu + (1)^{n-1} \lambda) & \mu (\lambda + \mu)^{n-1} ((-1)^n \mu + (-1)^{n-1} (\lambda)) \\ \lambda (\lambda + \mu)^{n-1} (-1)^{n-1} (\mu + (-1)^n \lambda) & \lambda (\lambda + \mu)^{n-1} ((-1)^{n-1} \mu + (-1)^n (\lambda)) \end{bmatrix}
\]

\[
= \begin{bmatrix} \mu (\lambda + \mu)^{n-1} (-1)^{n+1} \mu + \lambda & \mu (\lambda + \mu)^{n-1} (-1)^n (\mu + \lambda) \\ \lambda (\lambda + \mu)^{n-1} (-1)^n (\mu + \lambda) & \lambda (\lambda + \mu)^{n-1} (-1)^{n+1} (\mu + \lambda) \end{bmatrix}
\]

and so the given form for \( G^n \) is true for all \( n \geq 1 \).

(b) (5 marks) Noting that \((tG)^0/0! = G^0 = I\), the \( 2 \times 2 \) identity matrix, we have

\[
p_{11}(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n (-1)^n \mu (\lambda + \mu)^{n-1}}{n!} = 1 + \frac{\mu}{\lambda + \mu} \sum_{n=1}^{\infty} \frac{(-t(\lambda + \mu))_n}{n!} = 1 + \frac{\mu}{\lambda + \mu} \left( e^{-t(\lambda + \mu)} - 1 \right) = \frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-t(\lambda + \mu)}.
\]

Similarly,

\[
p_{22}(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n (-1)^n \lambda (\lambda + \mu)^{n-1}}{n!} = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-t(\lambda + \mu)}
\]

\[
p_{12}(t) = \frac{-\mu}{\lambda + \mu} \sum_{n=1}^{\infty} \frac{(-t(\lambda + \mu))_n}{n!} = \frac{\mu}{\lambda + \mu} - \frac{\mu}{\lambda + \mu} e^{-t(\lambda + \mu)}
\]

\[
p_{21}(t) = \frac{-\lambda}{\lambda + \mu} \sum_{n=1}^{\infty} \frac{(-t(\lambda + \mu))_n}{n!} = \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} e^{-t(\lambda + \mu)}.
\]

Thus,

\[
P(t) = \frac{1}{\lambda + \mu} \begin{bmatrix} \lambda + \mu e^{-t(\lambda + \mu)} & \mu - \mu e^{-t(\lambda + \mu)} \\ \lambda - \lambda e^{-t(\lambda + \mu)} & \mu + \lambda e^{-t(\lambda + \mu)} \end{bmatrix}.
\]

Clearly, \( P(0) = I \), and direct differentiation yields

\[
P'(t) = \frac{1}{\lambda + \mu} \begin{bmatrix} -\mu (\lambda + \mu) e^{-t(\lambda + \mu)} & \mu (\lambda + \mu) e^{-t(\lambda + \mu)} \\ \lambda (\lambda + \mu) e^{-t(\lambda + \mu)} & -\lambda (\lambda + \mu) e^{-t(\lambda + \mu)} \end{bmatrix}.
\]
On the other hand, it is straightforward to check that

\[ GP(t) = \frac{1}{\lambda + \mu} \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix} \begin{bmatrix} \lambda + \mu e^{-t(\lambda + \mu)} & \mu e^{-t(\lambda + \mu)} \\ \lambda e^{-t(\lambda + \mu)} & \mu + \lambda e^{-t(\lambda + \mu)} \end{bmatrix} \]

\[ = \frac{1}{\lambda + \mu} \begin{bmatrix} -\mu(\lambda + \mu)e^{-t(\lambda + \mu)} & \mu(\lambda + \mu)e^{-t(\lambda + \mu)} \\ \lambda(\lambda + \mu)e^{-t(\lambda + \mu)} & -\lambda(\lambda + \mu)e^{-t(\lambda + \mu)} \end{bmatrix} = P'(t). \]

Thus, \( P(t) \) satisfies the backward equations \( P'(t) = GP(t) \) with initial condition \( P(0) = I \). Finally, it is easy to see that

\[ \lim_{t \to \infty} P(t) = \frac{1}{\lambda + \mu} \begin{bmatrix} \lambda & \mu \\ \lambda & \mu \end{bmatrix}. \]

On the other hand, the stationary distribution \( \pi = (\pi_1, \pi_2) \) satisfies the global balance equations \( \pi G = 0 \) and \( \pi_1 + \pi_2 = 1 \), or

\[ \mu \pi_1 = \lambda \pi_2 \quad \text{and} \quad \pi_1 + \pi_2 = 1. \]

Solving the above gives \( \pi_1(1 + \mu/\lambda) = 1 \), or \( \pi_1 = \lambda/(\lambda + \mu) \), which then gives \( \pi_2 = \mu/(\lambda + \mu) \). Thus, we see that

\[ \lim_{t \to \infty} P(t) = \begin{bmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{bmatrix}, \]

as expected.

5. From Sheet. (6 marks)

(a) (3 marks) The exponential rates for the \( M/M/1 \) queue are \( q_{i,i+1} = \lambda \) for \( i \geq 0 \) and \( q_{i,i-1} = \mu \) for \( i \geq 1 \). All other rates are equal to zero. The local balance equations are

\[ \pi_i \lambda = \pi_{i+1} \mu \]

for \( i \geq 0 \), giving

\[ \pi_{i+1} = \frac{\lambda}{\mu} \pi_i = \left( \frac{\lambda}{\mu} \right)^2 \pi_{i-1} = \ldots = \left( \frac{\lambda}{\mu} \right)^{i+1} \pi_0. \]

Solving for \( \pi_0 \) using the normalization constraint we have

\[ 1 = \pi_0 \left( 1 + \frac{\lambda}{\mu} + \left( \frac{\lambda}{\mu} \right)^2 + \ldots \right) \]

\[ = \pi_0 \sum_{i=0}^{\infty} \left( \frac{\lambda}{\mu} \right)^i = \frac{\pi_0}{1 - \lambda/\mu}, \]

or \( \pi_0 = 1 - \lambda/\mu \). Then \( \pi_i = \left( \frac{\lambda}{\mu} \right)^i \left( 1 - \frac{\lambda}{\mu} \right) \), for \( i \geq 1 \).
(b) (3 marks) This is similar to part(a) except that the exponential rates are now $q_{i,i+1} = \lambda \times \frac{1}{i+1}$ for $i \geq 0$ and $q_{i,i-1} = \mu$ for $i \geq 1$. The local balance equations are

$$\pi_i \frac{\lambda}{i+1} = \pi_{i+1} \mu$$

for $i \geq 0$, giving

$$\pi_{i+1} = \frac{1}{i+1} \frac{\lambda}{\mu} \pi_i = \frac{1}{(i+1)!} \left(\frac{\lambda}{\mu}\right)^i \pi_{i-1} = \ldots = \frac{1}{(i+1)!} \left(\frac{\lambda}{\mu}\right)^{i+1} \pi_0.$$ 

Solving for $\pi_0$ using the normalization constraint we have

$$1 = \pi_0 \sum_{i=0}^{\infty} \frac{(\lambda/\mu)^i}{i!} = e^{\lambda/\mu},$$

or $\pi_0 = e^{-\lambda/\mu}$. Then $\pi_i = \frac{(\lambda/\mu)^i}{i!} e^{-(\lambda/\mu)}$ for $i \geq 0$ (i.e., $\pi$ is the Poisson($\lambda/\mu$) distribution).