Applied Stochastic Processes

Midterm Exam Practice Problems

1. Let \( \{X_n : n \geq 0\} \) be a Markov chain with state space \( S = \{1, 2, 3, 4, 5, 6\} \) and transition probability matrix

\[
P = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 0 & 0 & 1/2 & 0 & 1/3 \\
2 & 0 & 0 & 0 & 1 & 0 \\
3 & 1/3 & 1/3 & 0 & 0 & 1/3 \\
4 & 0 & 0 & 0 & 1 & 0 \\
5 & 0 & 1 & 0 & 0 & 0 \\
6 & 1/2 & 0 & 1/2 & 0 & 0 
\end{bmatrix}
\]

(a) Identify the equivalence classes and for each class, state whether it is transient or recurrent, and give the period of the class.

(b) It is clear that once the chain enters state 4, it will stay there forever. Starting in state 1, find the probability that the chain ultimately ends up in state 4. \textit{Hint:} Let \( p_1 \) denote this probability. Define \( p_3 \) and \( p_6 \) similarly and, by conditioning on the first move, write a set of 3 equations involving \( p_1, p_3, \) and \( p_6. \)

2. Consider the simple random walk with the probability of making an up move given by \( p. \) Let \( r \) be a given positive integer and let \( T \) denote the first time the walk finishes a string of \( r \) consecutive up moves. For \( k > r \) argue that

\[
P(T = k) = P(T > k - r - 1)qp^r,
\]

where \( q = 1 - p, \) and use this to show that

\[
G(s) = \frac{p^r s^r - p^{r+1} s^{r+1}}{1 - s + qp^r s^{r+1}},
\]

where \( G \) is the generating function of the distribution of \( T. \)