MTHE/STAT455, STAT855, Stochastic Processes

Midterm Exam

Instructions:

(a) The exam is closed book. No books are allowed. You may use one 8.5 $\times$ 11 inch sheet of notes and a calculator.

(b) There are 3 questions. Stat 855 students must do all of the problems. Mthe/Stat 455 students must do 2 of questions 1, 2 and 3, and if you answer all three questions, you must specify which two you want graded (the default if you do not specify is questions 1 and 2).

(c) Each question is worth 15 marks for a total of 30 marks (Mthe/Stat455) or 45 marks (Stat855)

(d) Show all your work. Partial credit is given.

(e) Read the hints!

(f) You have 120 minutes. Good luck!
1. (15 marks) Consider a square, where proceeding in a clockwise fashion, the vertices are labelled 1, 2, 3, 4. A particle performs a symmetric random walk on the vertices of the square (i.e., when it is at a given vertex it chooses one of its two neighbouring vertices with equal probability, then moves next to the chosen vertex). If the particle starts at vertex 1, find

(a) (7 marks) the expected time to first reach vertex 2, the expected time to first reach vertex 3, and the expected time to return to vertex 1.

(b) (4 marks) the expected time until vertices 2 and 3 have both been reached.

(c) (4 marks) the expected time until vertices 2, 3 and 4 have all been reached.

**Hint:** For appropriate subsets \( B \) of the vertex labels, define \( M_{i,B} \) to be the expected number of moves until all the vertices in \( B \) are visited, starting at vertex \( i \). These are all the unknowns you should need to define to answer parts (a)-(c).

2. (15 marks) For each of the following descriptions of a Markov chain \( \{X_n : n \geq 0\} \) write down the state space, the transition probabilities, the period of each state, and for each state whether it is recurrent or transient.

(a) (5 marks) Consider the following movement of two particles \( A \) and \( B \) on the nonnegative integers. Suppose particle \( A \) is initially at 0 and particle \( B \) is initially at \( k \), where \( k \geq 0 \). At the end of each time unit, if they are not at the same position, particle \( A \) moves one unit to the right with probability \( \alpha \) and remains where it is with probability \( 1 - \alpha \), while particle \( B \) moves one unit to the right with probability \( \beta \) and remains where it is with probability \( 1 - \beta \). Suppose \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \). Whenever \( A \) and \( B \) are at the same position, we say that \( A \) has found \( B \) and they stay in that position forever. Let \( X_n = B's \) position minus \( A's \) position after the end of the \( n \)th time unit.

(b) (5 marks) Consider the following movement of two particles \( A \) and \( B \) on \( N \) points that are equally spaced on a circle. The points are labelled 1 to \( N \) clockwise starting from point 1. Suppose particle \( A \) is initially at point 1 and particle \( B \) is initially at point \( k \), where \( 1 \leq k \leq N \). At the end of each time unit, if they are not at the same point, particle \( A \) moves one unit clockwise with probability \( \alpha \) and remains where it is with probability \( 1 - \alpha \), while particle \( B \) moves one unit clockwise with probability \( \beta \) and remains where it is with probability \( 1 - \beta \). Suppose \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \). Whenever \( A \) and \( B \) are at the same point, we say that \( A \) has found \( B \) and they stay at that point forever. Let \( X_n \) be the number of clockwise moves that particle \( A \) would have to make to get to the point where \( B \) is, after the end of the \( n \)th time unit.

(c) (5 marks) \( X_n = Y_n - Z_n \), where \( Y = \{Y_n : n \geq 0\} \) and \( Z = \{Z_n : n \geq 0\} \) are two independent simple random walks, each with the same probability of moving up given by \( p \), where \( 0 < p < 1 \).
3. (15 marks) Let \( \{X_n : n \geq 0\} \) be a Galton-Watson branching process starting at \( X_0 = 1 \) and with Geometric family size distribution

\[
P(X_1 = k) = \left( \frac{1}{2} \right)^{k+1} \quad \text{for } k = 0, 1, 2, \ldots\]

(a) (7 marks) Find the generating function of the family size distribution and, by induction, show that the generating function of \( X_n \) is given by

\[
G_n(s) = \frac{n - (n - 1)s}{n + 1 - ns}
\]

for \( n \geq 1 \).

(b) (3 marks) Find the probability of ultimate extinction.

(c) (5 marks) Let \( V_1 \) be the total number of generations of size 1. Find \( E[V_1] \). Hint: Use part(a). You may use the fact that \( G_n'(0) = P(X_n = 1) \) and \( \sum_{n=1}^{\infty} (1/n^2) = \pi^2/6 \).