1. (15 marks)

(a) (7 marks) For $3 \leq j \leq n$, starting at the $j$th best point we condition on the rank $R$ of the point we jump to next. By assumption, we have $P(R = k) = 1/(j - 1)$ for $k = 1, \ldots, j - 1$. Thus, we have

$$r_j = \sum_{k=1}^{j-1} (1 + r_k) \frac{1}{j - 1} = 1 + \frac{1}{j - 1} \sum_{k=1}^{j-1} r_k.$$ 

To solve this first multiply through by $j - 1$, giving

$$(j - 1)r_j = j - 1 + \sum_{k=1}^{j-1} r_k.$$ 

Next, write the corresponding equation for $r_{j+1}$ and subtract from it the equation for $r_j$, giving

$$jr_{j+1} - (j - 1)r_j = j + \sum_{k=1}^{j} r_k - (j - 1) - \sum_{k=1}^{j-1} r_k = 1 + r_j.$$ 

Thus, $jr_{j+1} - jr_j = 1$, or

$$r_{j+1} = r_j + \frac{1}{j} = r_{j-1} + \frac{1}{j - 1} + \frac{1}{j} = \ldots = r_1 + \sum_{k=1}^{j} \frac{1}{k} = \sum_{k=1}^{j} \frac{1}{k},$$ 

since $r_1 = 0$. Thus, $r_j = \sum_{k=1}^{j-1} k^{-1}$, for $j = 2, \ldots, n$.

(b) (8 marks) Note that $p_1 = 1$ and $p_2 = 0$. Let $Y$ indicate the first pair to combine: for $k = 1, \ldots, n - 1$, $Y = k$ means the pair $(m_k, m_{k+1})$ was the first to combine. We condition on $Y$. If $Y = 1$ then molecule $m_1$ will not be isolated, if $Y = 2$ then $m_1$ will remain isolated for sure, and if $Y = 3$ then $m_1$ will not be isolated. For $k = 4, \ldots, n - 1$, if $Y = k$ then we can assume we just started with molecules $m_1, \ldots, m_{k-1}$. Since $P(Y = k) = 1/(n - 1)$ for all $k$, we have

$$p_n = \frac{1}{n - 1} \left(0 + 1 + 0 + \sum_{k=4}^{n-1} p_{k-1}\right) = \frac{1}{n - 1} \sum_{k=1}^{n-2} p_k.$$
Therefore, \((n - 1)p_n = \sum_{k=1}^{n-2} p_k\). Also, \(np_{n+1} = \sum_{k=1}^{n-1} p_k\). Subtracting, we have

\[
np_{n+1} - (n - 1)p_n = p_{n-1} \Rightarrow n(p_{n+1} - p_n) = -(p_n - p_{n-1})
\]

\[
\Rightarrow p_{n+1} - p_n = -\frac{1}{n}(p_n - p_{n-1}).
\]

So

\[
p_{n+1} - p_n = -\frac{1}{n}(p_n - p_{n-1})
\]

\[
= (-1)^2 \frac{1}{n(n-1)}(p_{n-1} - p_{n-2})
\]

\[
\vdots
\]

\[
= (-1)^{n-2} \frac{1}{n(n-1) \times \ldots \times 3}(p_3 - p_2)
\]

\[
= \frac{2(-1)^{n-2}}{n!} p_3,
\]

since \(p_2 = 0\). But, direct calculation yields \(p_3 = 1/2\) and so we have \(p_{n+1} - p_n = (-1)^{n-2}/n! = (-1)^n/n!\) for \(n \geq 3\). Thus,

\[
p_{n+1} = p_n + \frac{(-1)^n}{n!}
\]

\[
= p_{n-1} + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}
\]

\[
\vdots
\]

\[
= p_3 + \sum_{r=3}^{n} \frac{(-1)^r}{r!}
\]

\[
= \sum_{r=2}^{n} \frac{(-1)^r}{r!}
\]

\[
= \sum_{r=0}^{n} \frac{(-1)^r}{r!}.
\]

Thus, \(p_n = \sum_{r=0}^{n-1} \frac{(-1)^r}{r!}\) as desired. One can check that the formula works for \(n = 1\) and \(n = 2\) as well.
2. (15 marks)

(a) (2 marks) From the diagram, we get the transition probability matrix as

\[
P = \begin{bmatrix}
0 & 0.5 & 0.5 & 0 & 0 \\
0.1 & 0 & 0.5 & 0 & 0.4 \\
0.1 & 0.1 & 0.8 & 0 & 0 \\
0 & 0 & 0.1 & 0.8 & 1 \\
0 & 0.1 & 0.4 & 0 & 0.5 \\
\end{bmatrix}.
\]

(b) (3 marks) Once Reilly leaves state 4 he will never return. Starting in state 4, the total time (in minutes) that he spends in state 4 is \(5X\), where \(X\) is geometrically distributed with parameter \(p = 0.2\). Thus, the total expected time Reilly spends in state 4 is \(5 \frac{1}{0.2} = 25\) minutes.

(c) (7 marks) State 4 is a transient state and all other states are positive recurrent. Since we start in state 5 we will stay in the set of states \(\{1, 2, 3, 5\}\) forever. We need to compute the stationary distribution \(\pi = (\pi_1, \pi_2, \pi_3, \pi_5)\) on this set of states. Then the mean time (in minutes) to return to state 5, starting in state 5, is \(\mu_5 = 5/\pi_5\). We first remove row and column 4 from the transition matrix then choose 3 of the resulting balance equations and the normalization constraint to compute \(\pi\). The 3 simplest balance equations and the normalization constraint are

\[
\begin{align*}
\pi_1 &= 0.1\pi_2 + 0.1\pi_3 \\
\pi_2 &= 0.5\pi_1 + 0.1\pi_3 + 0.1\pi_5 \\
\pi_5 &= 0.4\pi_2 + 0.5\pi_5 \\
1 &= \pi_1 + \pi_2 + \pi_3 + \pi_5
\end{align*}
\]

Eq.(3) gives \(\pi_5 = 0.8\pi_2\). Plugging this into Eq.(2) gives

\[
0.92\pi_2 = 0.5\pi_1 + 0.1\pi_3 \Rightarrow 0.1\pi_3 = 0.92\pi_2 - 0.5\pi_1.
\]

Plugging this into Eq.(1) gives \(1.5\pi_1 = 1.02\pi_2 \Rightarrow \pi_1 = 0.68\pi_2\). Plugging this into Eq.(5) gives \(0.1\pi_3 = 0.58\pi_2 \Rightarrow \pi_3 = 5.8\pi_2\). The normalization constraint (Eq.(4)) gives

\[
1 = \pi_2(0.68 + 1 + 5.8 + 0.8) = 8.28\pi_2 \Rightarrow \pi_2 = 1/8.28.
\]

Thus, \(\pi_5 = 0.8\pi_2 = 0.8/8.28\) and

\[
\mu_5 = \frac{5}{\pi_5} = \frac{5 \times 8.28}{0.8} = 51.75 \text{ minutes.}
\]
(d) (3 marks) We wish to compute $p_{33}(2)$, the 2-step transition probability from state 3 to state 3. This is given by the $(3, 3)$ entry of $P(2) = P^2$. This is given by

$$p_{33}(2) = (0.5)(0.1) + (0.5)(0.1) + (0.8)(0.8) = 0.74.$$ 

3. (15 marks)

(a) (3 marks) Let type 1 events be those for which the CF is greater than 2. Then the probability that a given event is type 1 is $p_1 = e^{-2c}$, independent of the time of the event. The time until the first type 1 event is exponential with parameter $\lambda p_1$ and mean $1/(\lambda p_1) = 1/(\lambda e^{-2c})$.

(b) (6 marks) One may directly use results for the filtered Poisson process here. Alternatively, we can use more basic properties of the Poisson process to compute the desired expectation. First, condition on $N(t)$. Given $N(t) = n$, we have

$$E[CE(t) \mid N(t) = n] = E\left[\sum_{i=1}^{n} CF_i e^{-r(t-S_i)} \mid N(t) = n\right]$$

$$= \sum_{i=1}^{n} E[CF_i e^{-r(t-S_i)} \mid N(t) = n]$$

$$= \sum_{i=1}^{n} E[CF_i e^{-r(t-U_i)}]$$

$$= \sum_{i=1}^{n} E[CF_i]E[e^{-r(t-U_i)}]$$

$$= \frac{n}{ct} \int_0^t e^{-r(t-s)} ds$$

$$= \frac{n}{ct} e^{-r(t-s)} \bigg|_0^t$$

$$= \frac{n}{ct} (1 - e^{-rt})$$

Unconditioning, we have

$$E[CE(t)] = \frac{E[N(t)](1 - e^{-rt})}{rc} = \frac{\lambda t(1 - e^{-rt})}{rc} = \frac{\lambda (1 - e^{-rt})}{rc}.$$ 

(c) (6 marks) Let $A$ denote the event that a Second Age of Cuteness comes. Conditioned on $T = t$ and $N(t) = n$, event $A$ occurs if and only if $\max(X_1, \ldots, X_n) > a$,
where \( X_1, \ldots, X_n \) are independent and identically distributed exponential random variables with parameter \( c \). We have

\[
P(A \mid T = t, N(t) = n) = P(\max(X_1, \ldots, X_n) > a)
\]

\[
= 1 - P(\max(X_1, \ldots, X_n) \leq a)
\]

\[
= 1 - P(X_1 \leq a, \ldots, X_n \leq a)
\]

\[
= 1 - (1 - e^{-ca})^n.
\]

Unconditioning, we have

\[
P(A) = \int_0^\infty \sum_{n=0}^\infty \left(1 - (1 - e^{-ca})^n\right) \frac{(\lambda t)^n}{n!} e^{-\lambda t} \mu e^{-\mu t} dt
\]

\[
= \int_0^\infty \mu e^{-(\mu+\lambda)t} \left(e^{\lambda t} - e^{\lambda t(1-e^{-ca})}\right) dt
\]

\[
= \int_0^\infty \left(\mu e^{-\mu t} - \mu e^{-(\mu+\lambda)e^{-ca}t}\right) dt
\]

\[
= 1 - \frac{\mu}{\mu + \lambda e^{-ca}}
\]

\[
= \frac{\lambda e^{-ca}}{\mu + \lambda e^{-ca}}.
\]

4. (15 marks)

(a) (3 marks) True. The (discrete) uniform distribution, \( \pi_i = 1/M \), satisfies the global balance equations.

(b) (3 marks) False. For example, let there be \( M = 3 \) states and define \( G \), and the corresponding \( P \), by

\[
G = \begin{bmatrix}
-2\lambda & \lambda & \lambda \\
2\lambda & -3\lambda & \lambda \\
0 & 2\lambda & -2\lambda
\end{bmatrix}
\]

and

\[
P = \begin{bmatrix}
0 & 1/2 & 1/2 \\
2/3 & 0 & 1/3 \\
0 & 1 & 0
\end{bmatrix},
\]

where \( \lambda > 0 \) is fixed. We see that the column sums of \( G \) are all equal to 0 but the column sums of \( P \) are not equal to 1.

(c) (3 marks) True. The transition matrix for \( Y \) is \( P(1) \), where \( P(t) \) is the matrix of transition probability functions for \( X \). That \( \pi = \pi P(1) \) is true follows from the definition of \( \pi \).
(d) (3 marks) True. If state \( i \) is recurrent then it is recurrent in the \( X \) chain; i.e., there is a finite sequence of transitions from \( i \) back to \( i \) that has positive probability. Moreover (with the regularity conditions), the sum of the exponential holding times in this sequence is less than 1 with positive probability; that is, \( p_{ii}(1) > 0 \), where \( p_{ii}(t) \) is the \((i, i)\)th transition probability function for the \( X \) chain. But \( p_{ii}(1) \) is the transition probability from \( i \) back to \( i \) in the \( Y \) chain (see part (c)). Therefore, state \( i \) is aperiodic.

(e) (3 marks) False. For example, let
\[
G = \begin{bmatrix}
-\lambda & \lambda \\
\lambda & -\lambda
\end{bmatrix},
\]
where \( \lambda > 0 \). The embedded jump chain simply jumps back and forth between states 1 and 2 deterministically and has period 2, though both states are recurrent.

(f) (2 bonus marks) True. Recall that the embedded jump chain has stationary distribution \( \psi \), where \( \psi_i \) is proportional to \( v_i \pi_i \). If \( v_i = v \) are all equal then \( \psi_i \) is proportional to \( \pi_i \) and so must equal \( \pi_i \).

5. (15 marks)

(a) (8 marks) We have that \( N(t) \) is the number of points whose squared distances from the origin are less than or equal to \( t \). We need to show the following:

(i) (3 marks) \( N(t) \) is Poisson distributed with parameter \( \lambda \pi t \): \( N(t) \) is the number of points in the circle of radius \( \sqrt{t} \) centred at the origin. Let \( C(t) \) denote this circle. By definition, this number is Poisson distributed with parameter
\[
\lambda \times \text{area}(C(t)) = \lambda \pi (\sqrt{t})^2 = \lambda \pi t.
\]

(ii) (2 marks) \( N \) has independent increments: Let \( (t_1, t_2) \) and \( (t_3, t_4) \) be disjoint, nonempty, intervals. Then \( N(t_2) - N(t_1) \) is the number of points in \( C(t_1, t_2) \) and \( N(t_4) - N(t_3) \) is the number of points in \( C(t_3, t_4) \), where, for \( x < y \), \( C(x, y) \) is the number of points in the annulus (the doughnut) formed by taking the circle of radius \( \sqrt{y} \) centred at the origin and subtracting from it the circle of radius \( \sqrt{x} \) centred at the origin. Since \( C(t_1, t_2) \) and \( C(t_3, t_4) \) are disjoint, it follows by definition that the increments \( N(t_2) - N(t_1) \) and \( N(t_4) - N(t_3) \) are independent.
(iii) (3 marks) $N$ has stationary increments: Let $(t_1, t_2)$ and $(t_3, t_4)$ be two intervals of equal length. Using the notation $C(x, y)$ of part (ii), we have that $N(t_2) - N(t_1)$ is Poisson distributed with parameter

$$\lambda \times \text{area}(C(t_1, t_2)) = \lambda(\pi t_2 - \pi t_1) = \lambda \pi (t_2 - t_1).$$

Similarly, the distribution of $N(t_4) - N(t_3)$ is Poisson with parameter $\lambda \pi (t_4 - t_3)$. Since $t_2 - t_1 = t_4 - t_3$ the increments $N(t_2) - N(t_1)$ and $N(t_4) - N(t_3)$ are identically distributed.

(b) (7 marks) Let $A(x)$ denote the circle of radius $x$ centred at the origin. From the hint we have, for $x > 0$,

$$P(R_k \leq x) = P(A(x) \text{ contains at least } k \text{ points})$$

$$= \sum_{j=k}^{\infty} P(A(x) \text{ contains exactly } j \text{ points})$$

$$= \sum_{j=k}^{\infty} \frac{(\lambda \pi x^2)^j}{j!} e^{-\lambda \pi x^2}.$$

We now differentiate to obtain, for $x > 0$,

$$f(x) = \sum_{j=k}^{\infty} \left( \frac{j(\lambda \pi x^2)^{j-1} 2 \lambda \pi x}{j!} e^{-\lambda \pi x^2} - 2 \lambda \pi x \frac{(\lambda \pi x^2)^j}{j!} e^{-\lambda \pi x^2} \right)$$

$$= 2 \lambda \pi x e^{-\lambda \pi x^2} \left( \sum_{j=k-1}^{\infty} \frac{(\lambda \pi x^2)^j}{j!} - \sum_{j=k}^{\infty} \frac{(\lambda \pi x^2)^j}{j!} \right)$$

$$= 2 \lambda \pi x e^{-\lambda \pi x^2} \frac{(\lambda \pi x^2)^{k-1}}{(k-1)!},$$

as desired. Clearly, $f(x) = 0$ for $x \leq 0$.

(c) (3 bonus marks) Let $X$ denote the area of the largest circle centred at the origin containing no points. Then, for $x > 0$, $X > x$ if and only if the circle of radius $\sqrt{x/\pi}$ centred at the origin contains no points. Since the number of points in this circle is Poisson distributed with parameter $\lambda \pi \sqrt{x/\pi^2} = \lambda x$, we have $P(X > x) = e^{-\lambda x}$, which shows that $X$ is exponentially distributed with parameter $\lambda$. 