Instructions:

• “Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.”

• “The candidate is urged to submit with the answer paper a clear statement of any assumptions made if doubt exists as to the interpretation of any question that requires a written answer.”

• This examination is THREE HOURS in length. It is closed-book – no books, notes, or any other resource material is allowed except as indicated in the next item.

• A standard non-communicating calculator without text storage capabilities and any notes on a single $8\frac{1}{2} \times 11$ inch sheet of paper (both sides) are allowed.

• Please answer all questions in the booklets provided. Put your student number on the front of all answer booklets and number the answer booklets if more than one answer booklet is used.

• There are 5 questions. Stat455 students must do questions 1 to 4. Stat855 students must do all 5 questions.

• Each question is worth 15 marks for a possible total of 60 for Stat455 students and 75 for Stat855 students.

• Show all your work. You may receive partial credit if you get an answer wrong but show your work, whereas you will receive no credit if you get an answer wrong and do not show your work.
1. (Total 15 marks) Let \( \{X_n\}_{n=0}^{\infty} \) be a discrete time Markov chain with state space \( S = \{0, 1, 2\} \) and initial distribution \( (1/3, 1/3, 1/3) \). Consider the three transition probability matrices

\[
(i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad (ii) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.4 & 0.6 \\ 0 & 0.7 & 0.3 \end{bmatrix}, \quad (iii) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.2 & 0.2 \end{bmatrix}
\]

For each of the transition matrices above, do the following:

(a) ( marks) Specify the classes, and for each class specify whether the class is positive recurrent, null recurrent, or transient, and also the period of the class.

(b) ( marks) Let \( P_j(n) = P(X_n = j) \), for \( j = 0, 1, 2 \) and \( n \geq 0 \). By conditioning on the initial state, find \( \lim_{n \to \infty} P_j(n) \), for \( j = 0, 1, 2 \).

2. (Total 15 marks) Let \( \{X_n : n \geq 0\} \) be a Galton-Watson branching process starting at \( X_0 = 1 \) and with Geometric family size distribution

\[
P(X_1 = k) = \left(\frac{1}{2}\right)^{k+1} \quad \text{for } k = 0, 1, 2, \ldots
\]

(a) (8 marks) Find the generating function of the family size distribution and, by induction, show that the generating function of \( X_n \) is given by

\[
G_n(s) = \frac{n - (n - 1)s}{n + 1 - ns}
\]

for \( n \geq 1 \).

(b) (3 marks) Find the probability of ultimate extinction.

(c) (4 marks) Let \( V_1 \) be the total number of generations of size 1. Find \( E[V_1] \). \( \text{Hint: Use part (a). } \) You may use the fact that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6 \).

3. (Total 15 marks)

(a) (6 marks) Consider a large container with a hole in the bottom, out of which water can pour at a constant rate of 1 litre per minute. Assume that at time \( t = 0 \), the container is empty. At each event time of a Poisson process with rate 1, exactly 1 litre of water is poured into the container. Assume that it takes no time to pour the water. Compute the expected amount of water in the container at time \( t = 1 \). \( \text{Hint: Condition on the number of events by time 1.} \)

(b) (9 marks) Consider the \( M/G/\infty \) queue in which customers arrive as a Poisson process with rate \( \lambda \). Upon arrival, each customer immediately goes into service then at the completion of service departs the system. Service times are independent and identically distributed random variables with distribution function \( G \).
Fix $t > 0$ and $r > 0$. Show that the number of departures that ever occur in the interval $(t, t + r]$ follows a Poisson distribution with parameter

$$
\lambda \int_t^{t+r} G(y) dy.
$$

*Hint:* Define an arrival at time $s$ to be a Type 1 arrival if its departure time is in the interval $(t, t + r]$.

4. (Total 15 marks) Suppose that $M$ songs, labelled $1, \ldots, M$, are stored in a computer’s memory in an ordered list. For example, if there are $M = 3$ songs and the ordering is $(3, 2, 1)$ then song 3 is in position 1, song 2 is in position 2 and song 1 is in position 3. Suppose that song $i$ is requested at the times of a Poisson process with rate $\alpha_i > 0$, for $i = 1, \ldots, M$. Whenever song $i$ is requested it exchanges its position in the list with the song ahead of it, unless it is already in position 1 in which case it stays in position 1. Thus, in the example above if song 2 is requested the new ordering becomes $(2, 3, 1)$. Let $X(t)$ denote the ordering at time $t$, for $t \geq 0$. Then \{ $X(t) : t \geq 0$ \} is a continuous-time Markov chain with state space $S$ equal to the set of all permutations of $\{1, \ldots, M\}$.

(a) (8 marks) Using the notation $i = (i_1, \ldots, i_M)$ for an arbitrary state and $j_k = (i_1, \ldots, i_{k-2}, i_k, i_{k-1}, i_{k+1}, \ldots, i_M)$ for the state obtained by exchanging components $k - 1$ and $k$ of state $i$, write the local balance equations for the stationary probabilities $\pi_i$, $i \in S$. Solve for $\pi_i$, $i \in S$, up to a normalizing constant.

(b) (7 marks) Repeat part (a), except now instead of assuming that whenever song $i$ is requested it exchanges its position in the list with the song ahead of it, it exchanges its position in the list with the song behind it, unless it is in position $M$ in which case it stays in position $M$. Thus, in the example if song 2 is requested the new ordering becomes $(3, 1, 2)$.

*5. (Total 15 marks) Consider the discrete time GI/D/1 queue, described as follows. There is a single server. At the beginning of the $n$th time period $Z_n$ customers arrive for service, where $Z_1, Z_2, \ldots$ are i.i.d. with common p.m.f. $f(k)$ on $\{0, 1, 2, \ldots\}$ and mean $\mu < 1$. Service times are exactly one time unit long. If customers arrive at the start of a time period to an empty system, one of them will go into service and the others (if any) will form a queue. If the system is nonempty then these customers will join the end of the queue (in some order). Let $X_n$ be the number of customers in the system (either in the queue or in service) in the middle of the $n$th time period. Then \{ $X_n : n \geq 0$ \} is a Markov chain with state space $S = \{0, 1, 2, \ldots\}$. Indeed, if $X_n > 0$, then we have $X_{n+1} = X_n - 1 + Z_{n+1}$, while if $X_n = 0$, then $X_{n+1} = Z_{n+1}$. Assume that a stationary distribution $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$ exists (it does) and let $G_\pi(s)$ be its
probability generating function:

\[ G_\pi(s) = \sum_{n=0}^{\infty} s^n \pi_n. \]

(a) (3 marks) Write down the transition probability matrix of \( \{X_n\} \).

(b) (3 marks) Write down the \( j \)th global balance equation for \( \pi \).

(c) (7 marks) Show that \( G_\pi(s) \) is given by

\[ G_\pi(s) = \frac{\pi_0 G_Z(s)(s-1)}{s - G_Z(s)}, \]

where \( G_Z(s) \) is the probability generating function of \( Z_1 \). \textit{Hint:} Multiply the equation from part(b) by \( s^j \), then sum over \( j \) and simplify.

(d) (2 marks) Use the boundary condition \( G_\pi(1) = 1 \) and l’Hopital’s rule to show that \( \pi_0 = 1 - \mu \).

Some distributions (for reference):

- Poisson(\( \lambda \)) pmf: \( f(k) = \frac{\lambda^k}{k!} e^{-\lambda} \), for \( k = 0, 1, 2, \ldots \)
  \( (E[X] = \lambda; \ Var(X) = \lambda) \).

- Exponential(\( \lambda \)) pdf: \( f(x) = \lambda e^{-\lambda x} \), for \( x \geq 0 \)
  \( (E[X] = 1/\lambda; \ Var(X) = 1/\lambda^2) \).

- Uniform(\( a, b \)) pdf: \( f(x) = \frac{1}{b-a} \), for \( a \leq x \leq b \)
  \( (E[X] = (a+b)/2; \ Var(X) = (b-a)^2/12) \).

HAPPY HOLIDAYS!!