1. (15 marks)

(a) 3 marks) **True.** Take the simple random walk with \( p \neq 1/2 \).

(b) 3 marks) **True.** If state \( k \) were recurrent then the vector \( \mathbf{\rho}(k) = (\rho_i(k))_{i \in S} \), where \( \rho_i(k) \) is the mean number of visits to state \( i \) during a sojourn from \( k \) back to \( k \), satisfies \( \mathbf{\rho}(k) = \mathbf{\rho}(k)\mathbf{P} \).

(c) 3 marks) **False.** Take \( S \) to be the integers and for each \( i \in S \) suppose that \( p_{i,i+1} > 0, p_{i,i-1} > 0, p_{i,i+2} > 0, \) and \( p_{i,i-2} > 0 \). Then from state \( i \) one can make the transitions \( i \to i+1 \to i \) so that \( p_{ii}(2) > 0 \). One can also make the transitions \( i \to i+2 \to i+1 \to i \) so that \( p_{ii}(3) > 0 \). Since the greatest common divisor of 2 and 3 is 1, the period must be 1.

(d) 3 marks) **False.** Take \( S \) to be the integers and for each \( i \in S \) suppose that \( p_{ii} = r \), where \( 0 < r < 1 \), \( p_{i,i+1} = (1-r)/2 \), and \( p_{i,i-1} = (1-r)/2 \). If a stationary distribution existed then this chain is time reversible because for any states \( i \) and \( j \) and any path from \( i \) to \( j \), the reverse path will have the same probability. So the stationary distribution must satisfy the local balance equations, which are

\[
\frac{1-r}{2} = \frac{1-r}{2}
\]

for all \( i \). These reduce to \( \pi_i = \pi_{i-1} \) for all \( i \). Thus, all components of \( \pi \) must be the same. But this is impossible since \( S \) is infinite.

(e) 3 marks) **False.** Take the simple, symmetric random walk \( (p = 1/2) \), or the counterexample from part(d), where \( \mathbf{P} \) is symmetric but no stationary distribution exists.

(f) 2 bonus marks) **True.** If \( X \) were time reversible with stationary distribution \( \pi \) and \( \mathbf{P} \) were symmetric then the local balance equations

\[
\pi_i p_{ij} = \pi_j p_{ji}
\]

would reduce to \( \pi_i = \pi_j \) for all \( i, j \in S \), since \( p_{ij} = p_{ji} \). That is, all components of \( \pi \) would have to be equal, which is impossible if \( S \) is infinite.
2. (15 marks) We check the local balance equations for all parts.

(a) (5 marks) The local balance equations are

\[ \pi_i \left( \frac{n}{2} \right)^{-1} = \pi_j \left( \frac{n}{2} \right)^{-1} \]

for all \( i, j \) such that \( j \) is equal to \( i \) except with two components interchanged. Thus, we get \( \pi_i = \pi_j \) for all \( i, j \in S \). Since \( S \) has \( n! \) elements, we get

\[ \pi_i = \frac{1}{n!} \text{ for all } i \in S. \]

(b) (5 marks) The local balance equations are

\[ \pi_i a_{i_k} = \pi_j a_{i_1} \]

for all \( i, j \) such that \( j \) is equal to \( i \) except with components \( k \) and 1 interchanged, \( k > 1 \). Positing a product form for \( \pi_i \), that is \( \pi_i = f_1(i_1) \times \ldots \times f_n(i_n) \), the local balance equation reduces to

\[ f_1(i_1) f_k(i_k) a_{i_k} = f_1(i_k) f_1(i_1) a_{i_1}. \]

From this we see we can take \( f_k(\cdot) \) to be constant and \( f_1(r) = a_r \), for \( r = 1, \ldots, n. \) This gives \( \pi_i \propto a_{i_1} \). Upon normalization we get that

\[ \pi_i = \frac{a_{i_1}}{(n-1)!} \text{ for all } i \in S. \]

(c) (5 marks) The local balance equations are

\[ \pi_i a_{i_k} = \pi_j a_{i_{k-2}} \]

for all \( i, j \) such that \( j \) is equal to \( i \) except with components \( k \) and \( k-2 \) interchanged, \( k > 2 \). Positing a product form for \( \pi_i \), that is \( \pi_i = f_1(i_1) \times \ldots \times f_n(i_n) \), the local balance equation reduces to

\[ f_{k-2}(i_{k-2}) f_k(i_k) a_{i_k} = f_{k-2}(i_k) f_{k-2}(i_{k-2}) a_{i_{k-2}}. \]

What works here is to take \( f_k(r) = \frac{1}{a_{r+2}} \). We can check that

\[ \frac{1}{a_{i_{k-2}}} \frac{1}{a_{i_k}^{k/2}} a_{i_k} = \frac{1}{a_{i_k}^{(k-2)/2}} a_{i_{k-2}}^{k/2} \]
holds. Thus,
\[ \pi_i = C \left( \frac{1}{a_1 a_2^2 \times \ldots \times a_n^n} \right)^{1/2} \text{ for all } i \in S, \]
where
\[ C = \left[ \sum_{i \in S} \left( \frac{1}{a_1 a_2^2 \times \ldots \times a_n^n} \right)^{1/2} \right]^{-1} \]
is the normalizing constant.

3. (15 marks)

(a) (2 marks) The number of events from both processes by time \( t \) is \( N_1(t) + N_2(t) \), which has a Poisson distribution with parameter \( (\lambda_1 + \lambda_2)t \). The probability that exactly 10 events occur in total from both processes in the interval \([0, 2)\) is therefore,
\[ P(N_1(2) + N_2(2) = 10) = \frac{(2(\lambda_1 + \lambda_2))^{10}}{10!} e^{-2(\lambda_1 + \lambda_2)}. \]

(b) (3 marks) We wish to compute \( P(N_1(1) = 1, N_2(2) - N_2(1) = 1, N(2) = 3) \), where \( N(t) = N_1(t) + N_2(t) \). Expressing this in terms of independent events, we have
\[
P(N_1(1) = 1, N_2(2) - N_2(1) = 1, N(2) = 3) = P(N_1(1) = 1, N_1(2) - N_1(1) = 1, N_2(2) - N_2(1) = 1) + P(N_1(1) = 1, N_1(2) - N_1(1) = 0, N_2(2) - N_2(1) = 1) = \lambda_1 e^{-\lambda_1} \lambda_1 e^{-\lambda_1} \lambda_2 e^{-\lambda_2} + \lambda_1 e^{-\lambda_1} \lambda_1 e^{-\lambda_1} \lambda_2 e^{-\lambda_2} = \lambda_1 \lambda_2 e^{-2(\lambda_1 + \lambda_2)}(\lambda_1 + \lambda_2),\]
where in the second equality we have used the fact that both of the processes \( \{N_1(t) : t \geq 0\} \) and \( \{N_2(t) : t \geq 0\} \) have stationary and independent increments, and are independent of one another.

(c) (5 marks) Let \( A \) denote the event that the first 10 events in the superposition process \( \{N(t) : t \geq 0\} \), where \( N(t) = N_1(t) + N_2(t) \), alternate between PP1 and PP2 events, starting with a PP1 event. We wish to compute
\[ P(A, N_1(2) = 5, N_2(2) = 5) = P(A \mid N_1(2) = 5, N_2(2) = 5) P(N_1(2) = 5, N_2(2) = 5). \]
Now given there were 5 PP1 events and 5 PP2 events in the interval \([0, 2)\) the times of these events are independent and each is uniformly distributed in
the interval [0, 2). By an ‘ordering’ of these 10 events we mean, for example, (1, 2, 2, 2, 1, 1, 2, 1, 1, 2), which indicates that the first event was from PP1, the next three were from PP2, and so on. The event A is the event that the ordering (1, 2, 1, 2, 1, 2, 1, 2, 1, 2) occurred. By symmetry, each of these orderings is equally likely. There are \( \binom{10}{5} \) such orderings, so we get that

\[
P(A \mid N_1(2) = 5, N_2(2) = 5) = \frac{1}{\binom{10}{5}} = \frac{5!5!}{10!},
\]

and thus

\[
P(A \mid N_1(2) = 5, N_2(2) = 5)P(N_1(2) = 5, N_2(2) = 5) = \frac{5!5!}{10!} \frac{(2\lambda_1)^5}{5!} e^{-2\lambda_1} \frac{(2\lambda_2)^5}{5!} e^{-2\lambda_2} = \frac{(4\lambda_1\lambda_2)^5}{10!} e^{-2(\lambda_1+\lambda_2)}.
\]

(d) (5 marks) Let

\[
X(t) = \begin{cases} 
1 & \text{if the light is bright at time } t \\
0 & \text{if the light is dim at time } t
\end{cases}
\]

Then \( \{X(t) : t \geq 0\} \) is a continuous time Markov chain with generator matrix

\[
G = \begin{bmatrix} 
-\lambda_1 & \lambda_1 \\
\lambda_2 & -\lambda_2
\end{bmatrix},
\]

where the first row corresponds to state 0 and the second row corresponds to state 1. The proportion of time that the light is bright is just \( \pi_1 \), the stationary probability of state 1. To compute \( \pi = (\pi_0, \pi_1) \), we have from the first global balance equation that \( \lambda_1\pi_0 = \lambda_2\pi_1 \). The normalization constraint gives \( \pi_0 = 1 - \pi_1 \) and plugging this into the preceding equation gives \( \lambda_1(1 - \pi_1) = \lambda_2\pi_1 \). Solving for \( \pi_1 \), we get

\[
\lambda_1 = (\lambda_1 + \lambda_2)\pi_1 \quad \text{or} \quad \pi_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\]
4. (15 marks)

(a) (4 marks) The generator matrix of the chain can be written as

\[
G = \begin{bmatrix}
-1 & 0 & 0 & \cdots & 0 & f(b_1) & f(b_2) & \cdots & f(b_s) \\
0 & -1 & 0 & \cdots & 0 & f(b_1) & f(b_2) & \cdots & f(b_s) \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & 0 & f(b_1) & f(b_2) & \cdots & f(b_s) \\
\end{bmatrix}
\]

where the rows (and columns) correspond to the states according to the ordering \(a_1, \ldots, a_r, b_1, \ldots, b_s\), and

\[
v_{b_j} = \sum_{i=1}^{r} q_{b_j,a_i}
\]

is the total rate out of state \(b_j\).

(b) (6 marks) For a state \(a_i \in A\), the global balance equation is

\[
\pi_{a_i} = \sum_{j=1}^{s} \pi_{b_j} q_{b_j,a_i}
\]  \hfill (1)

while for a state \(b_j \in B\) the global balance equation is

\[
v_{b_j} \pi_{b_j} = \sum_{i=1}^{r} \pi_{a_i} f(b_j) = f(b_j) \sum_{i=1}^{r} \pi_{a_i}.
\]  \hfill (2)

Letting \(\pi_U = \sum_{i=1}^{r} \pi_{a_i}\) denote the stationary probability that the process is in the set \(A\), from (2), we have

\[
\pi_{b_j} = \frac{f(b_j) \pi_U}{v_{b_j}}
\]  \hfill (3)

and plugging this back into (1) we get

\[
\pi_{a_i} = \pi_U \sum_{j=1}^{s} \frac{f(b_j) q_{b_j,a_i}}{v_{b_j}}.
\]  \hfill (4)
Using (3) and (4), the normalization constraint gives

$$1 = \pi_U \left( \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{f(b_j)q_{b_j,a_i}}{v_{b_j}} + \sum_{j=1}^{s} \frac{f(b_j)}{v_{b_j}} \right) = \pi_U \left( 1 + \sum_{j=1}^{s} \frac{f(b_j)}{v_{b_j}} \right)$$

which gives

$$\pi_U = \left( 1 + \sum_{j=1}^{s} \frac{f(b_j)}{v_{b_j}} \right)^{-1}$$

and from (3) we obtain

$$\pi_{b_j} = \left( 1 + \sum_{j=1}^{s} \frac{f(b_j)}{v_{b_j}} \right)^{-1} \frac{f(b_j)}{v_{b_j}} \quad \text{for } j = 1, \ldots, s.$$

(c) (5 marks) For a pair of states $a_i \in A$ and $b_j \in B$, the local balance equation is given by

$$\pi_{a_i} f(b_j) = \pi_{b_j} g(a_i)$$

and all (nontrivial) local balance equations look like this. Clearly,

$$\pi_{a_i} = K g(a_i) \quad \text{and} \quad \pi_{b_j} = K f(b_j)$$

will satisfy the local balance equations, where $K$ is the normalizing constant:

$$K = \left( \sum_{j=1}^{s} f(b_j) + \sum_{i=1}^{r} g(a_i) \right)^{-1} = \left( 1 + \sum_{i=1}^{r} g(a_i) \right)^{-1}.$$

5. (15 marks)

(a) (8 marks) Transitions occur at the times of a Poisson process with rate $v$; let $\{N(t) : t \geq 0\}$ denote this Poisson process. If $X(0) = i$ and $N(t) = n$ then $X(t)$ will equal $j$ with probability $p_{ij}(n)$, where $p_{ij}(n)$ is the $n$-step transition probability from state $i$ to state $j$ in the embedded discrete time Markov chain. Thus, conditioning on $N(t)$, we have

$$p_{ij}(t) = P(X(t) = j \mid X(0) = i) = \sum_{n=0}^{\infty} P(X(t) = j \mid N(t) = n, X(0) = i)P(N(t) = n \mid X(0) = i)$$

$$= \sum_{n=0}^{\infty} p_{ij}(n) \frac{(vt)^n}{n!} e^{-vt},$$
or in matrix form

$$[P(t)]_{ij} = \sum_{n=0}^{\infty} [P^n]_{ij} \frac{(vt)^n}{n!} e^{-vt} = \left[ \sum_{n=0}^{\infty} \frac{P^n}{n!} e^{-vt} \right]_{ij}$$

which is the desired result.

(b) (7 marks) Suppose the chain starts in state $i$. If $N$ denotes the total number of exponential holding times in state $i$, then $N$ follows a Geometric distribution with parameter $1 - f_i$, where $f_i$ is the probability that the embedded discrete time chain ever returns to state $i$ given that it starts in state $i$. That is,

$$P(N = k \mid X(0) = i) = f_i^{k-1}(1 - f_i) \quad \text{for } k = 1, 2, \ldots$$

If $T$ denotes the total time spent in state $i$ then conditioned on $N = k$, $T$ has a Gamma distribution with parameters $k$ and $v$. Thus, we may compute the cdf of $T$ given $X(0) = i$ by conditioning on $N$, giving

$$P(T \leq t \mid X(0) = i) = \sum_{k=1}^{\infty} P(T \leq t \mid X(0) = i, N = k) P(N = k \mid X(0) = i)$$

$$= \sum_{k=1}^{\infty} \int_0^t \frac{v^k}{(k-1)!} x^{k-1} e^{-vx} dx f_i^{k-1}(1 - f_i)$$

$$= (1 - f_i) v \int_0^t \left[ \sum_{k=1}^{\infty} \frac{(v f_i x)^{k-1}}{(k-1)!} \right] e^{-vx} dx$$

$$= (1 - f_i) v \int_0^t e^{vf_i x} e^{-vx} dx$$

$$= (1 - f_i) v \int_0^t e^{-(1 - f_i)vx} dx$$

$$= (1 - f_i) v \left[ \frac{e^{-(1 - f_i)vx}}{(1 - f_i)v} \right]_0^t = 1 - e^{-(1 - f_i)vt}.$$ 

But this is the cdf of an exponential distribution with parameter $(1 - f_i)v$, so the result is proven.